A

# TREATISE

ON

# HYDROSTATICS

AND

# HYDRODYNAMICS:

FOR

The Use of Students

IN THE UNIVERSITY.


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OF ST. JOHN'S COLLEGE.





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# PREFACE.

In a Treatise intended for the purposes of Academical instruction it is of the first importance that the more elementary propositions should be laid down, reference being had only to the simplest and most obvious methods of investigation.

The attainment of this object has in the following work been found in some degree incompatible with a strictly scientific arrangement of its parts.

The discussion of the general equations of Equilibrium evidently forms the legitimate basis of a theory of Hydrostatics. This discussion involves however the consideration of a point in space referred to three rectangular co-ordinates, and is, in its most general form, by no means essential to a further progress in the subject: it has therefore been referred to the end of the work, and that particular case of it in which the accelerating force is gravity, considered alone.

In the theory of the *motion of fluids*, a distinction has been made between the case *in which the velocity of every particle is the same as it passes through the same point in space*; and the more extended case of variable motion. Of the former a separate solution has been obtained; and

on the resulting formula the whole of the theory of Hydrodynamics has been made to depend. The manner in which this part of the subject has been treated is believed to be altogether new.

It is unnecessary here to enter further into the arrangement of the work. The reader is referred to a copious table of Contents which has been prefixed to it.

In conclusion the Author has to acknowledge his important obligations to his friend Mr. Challis, of Trinity College. He is indebted to that Gentleman for the Chapter (VII.) on the general Equations of the Motion of Fluids, and the Appendix (A) on the Oscillations of a cylindrical Column of Air. In the former of these papers, Mr. Challis has completely solved the general equation expressing the continuity of a moving fluid.

WEST MONKTON, near TAUNTON,
*March* 30, 1830.

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Page 4, line 14, *for* as *read* of.

— 13, line 23, omit *the* and insert it after *from* in line 26.

— 21, *for* Art. 32, *read* as follows.—Hence therefore the pressure on the surface, or on any portion of the surface of a vessel containing fluid, is equal to the weight of a prism of the fluid whose base is equal to that surface, and height to the perpendicular depth of its center of gravity.

— 40, line 5, *for* $DN = NP$ *read* $DM = MP$.

— 81, line 3, from the bottom, *for* equilibriums *read* equilibrium.

— 86, line 8, *for* them *read* then.

— 118, line 14, call this equation ($\pi$).

— 141, Art. 130, throughout this Article, *for* $h$ *read* $hg$.

— 151, line 17, *for* (Fig. 35,) *read* (Fig. 37.)

— 155, line 13, *for* (Fig. 37,) *read* (Fig. 33.)

— 168, line 6, *omit* (Fig. 40.)

— 192, line 4 from the bottom, *for* line *read* sine.

— 198, line 4 from the bottom. In the denominator, *for* 1 *read* $\frac{1}{c}$.

— 206, line 10, *for* $\sqrt{gh\frac{D_1}{D_2}}$, *read* $\sqrt{\text{h. l. } gh\frac{D_1}{D_2}}$.

# ELEMENTS

OF

# HYDROSTATICS.

## CHAP. I.

1. Fluids differ from solid bodies in the slighter adhesion of their parts, and the facility with which they are made to move among one another.

2. They are distinguished into incompressible or liquid, and elastic or aeriform.

Incompressible fluids may be made to assume an infinite variety of different forms, but retain always the same volume.

Elastic fluids vary at once in form and volume with any variation in the pressure they sustain, and return again to the same form and volume when the same circumstances of pressure are restored.

3. Force impressed on a solid, is effective only in the direction in which it is impressed, and is sustained by equal force impressed in an opposite direction. Applied to a fluid, it is effective in every direction, and is only to be sustained by forces applied to every point in the surface of that fluid.

4. Conceive a vessel (Fig. 1.) to contain a fluid, with the whole of whose surface it is accurately in contact; and let such forces be applied to the fluid (by means of pistons *P* and *Q* acting through apertures any where made in the vessel), as may sustain one another and keep the whole at rest. Now, an equilibrium being thus established, it is a manifest and distinctive property of fluids, that such a force may be taken as, applied to either of the pistons, will cause the other piston to ascend or require an additional force to keep it at rest. In what are termed *perfect fluids*, any force, however small, is sufficient thus to disturb the equilibrium: and in solids, no force, however great. Between these extremes, nature presents us with an infinite variety of bodies, which appear to approach more or less to a state of perfect fluidity, as they are more or less affected by that mutual attraction of their parts which is called cohesion, and is common to all material bodies.

5. Since the same pressure is manifestly produced on either of the pistons, by the force applied to the other, as though it formed part of the containing vessel, it follows that such a force may be applied to one portion of the surface of a fluid (wholly enclosed by the containing vessel) as to produce a certain pressure in every other portion of that surface. Now, in perfect fluids, the pressure thus produced on any portion of surface equal to that of the piston, is the same with the pressure on the piston itself.

This property of the *equal* distribution of fluid pressure, may be proved directly by experiment, and thus proved, is commonly taken for the basis of the theory of Hydrostatics.

It may, however, be deduced on mechanical principles, from the more evident and characteristic properties of fluids.

6. Conceive the pistons *P* and *Q* to act through tubes to which they are accurately fitted: and suppose the system to be in equilibrium as above. Now, let the force *P* be applied to one of the pistons, and let *P′* be that force which must be similarly applied to the other, in order to maintain the equilibrium.

Since no change takes place in the system from the application of the forces $P$ and $P'$, and that no resistance is opposed to the mutual action of these forces upon one another, by the intervening fluid, (any the slightest pressure being communicated from one piston to the other, by hypothesis,) it is clear that the equilibrium among the forces first impressed upon it remains; and, therefore, that $P$ and $P'$ are themselves in equilibrium.

The whole may, therefore, be considered as a machine, on which these forces sustain one another, and of which the distinguishing property is this, that whatever motion takes place in it, the fluid between the pistons will continue to occupy the same space. Let one of the pistons be slightly thrust down, then will the other be raised; and as much fluid as is displaced by the first piston will be forced into the tube which contains the other. If, therefore, $h$ and $-h'$* be the distances through which the motion of the pistons takes place, and $N$ and $N'$ transverse sections of the tubes,

$$Nh + N'h' = 0.$$

Also, by the principle of virtual velocities, since the forces $P$ and $P'$ are in equilibrium on the system

$$Ph + P'h' = 0;$$

$$\therefore \frac{P}{N} = \frac{P'}{N'} \ldots\ldots\ldots\ldots\ldots (a).$$

The same reasoning may be extended to any number of pressures, by comparing $P$ separately with each, and considering the rest as supplied by the sides of the vessel, thus

$$\frac{P}{N} = \frac{P'}{N'} = \frac{P''}{N''} = \frac{P'''}{N'''} = \&c. \ \&c.$$

The above equations hold for elastic as well as inelastic fluids; for, the pressure being the same, the density is not altered by any alteration in the position of the pistons.

* The negative sign is here taken, because the motions of the two pistons take place in opposite directions.

7. We shall give a second demonstration of this important theorem, more directly grounded on experiment.

Let $AB$ (Fig. 4.) be an uniform bent tube containing a fluid in equilibrium, whose surfaces are at $A$ and $B$. Let a force $P$ be applied to the surface at $A$, by means of a piston accurately fitting the interior of the tube. It will be found that an equal force $Q$ must be similarly applied at $B$, in order to preserve the equilibrium in the same position of the pistons. And, similarly, if by means of a weight or other force applied at either surface, the fluid be forced into any other position $A'B'$, and rest in that position; then, the whole being in equilibrium, if a force $P$ be applied at $A'$, an equal force $Q$ must be applied at $B'$ to *preserve* the equilibrium in the same position as the fluid.

Now, the forces, whatever they may have been, which acted upon the fluid before the application of $P$ and $Q$, were in equilibrium, since they held the system at rest; also this equilibrium remains after the application of those forces, since the position of the fluid is unaltered. The forces, other than $P$ and $Q$, being, therefore, in equilibrium; also the whole system of forces, including these, being in equilibrium, they are themselves in equilibrium. And the force $P$ applied to the surface $A'$, sustaining the equal force $Q$, applied to the surface $B'$, it follows that the pressure on either surface is propagated through the fluid to the other.

*If there be a force impressed on a given surface, in any portion of a fluid at rest, an equal pressure will thereby be generated on an equal surface in any other portion of the fluid.*

Let $ABCD$ (Fig. 2.) be a vessel filled with fluid. Let a piston be introduced at $Q$; and suppose a given force $Q$ to act upon it. The pressure $Q$ will generate in any other portion of the fluid an equal pressure on a surface $M$, equal to that of the piston. For, since the fluid is in equilibrium, if any of the parts of it be connected together, so as to become solid, the equilibrium will continue under the same circumstances with regard to the remainder, it being impossible that

an equilibrium, once established, should be destroyed, except by adding to the forces from which it has resulted, or taking away from them. Neither of which cases are involved in our supposition.

Let, then, every portion of the fluid be supposed to become solid, excepting only the uniform tube $QM$ extending from the piston $Q$ to the surface $M$: the equilibrium will, therefore, remain with regard to the fluid within the tube $QM$: and the pressure on $M$ will be the same as when the surrounding particles were in a fluid state.

Now, by the last article, the pressure communicated by $Q$* to $M$ is equal to $Q$: and $M$ is anywhere situated in the fluid: whence the truth of the proposition is apparent.

Let $N$ and $N'$ be surfaces taken anywhere in the fluid, of which let $M$, the area of the section of the piston $Q$, be a common measure. Also, let $M$ be exceeding small, so that $N$ and $N'$ may be considered as made up of elementary planes, each equal to $M$. Then will the pressure on each of these elementary planes be represented by $Q$, and their numbers in the two surfaces respectively, by $\frac{N}{M}$ and $\frac{N'}{M}$, putting, therefore, $P$ and $P'$ for the whole pressures sustained by the surfaces, we have

$$P = \frac{N}{M} \cdot Q,$$

$$P' = \frac{N'}{M} Q,$$

$$\frac{P}{P'} = \frac{N}{N'}.$$

---

* *The "pressure communicated by Q,"* it is to be observed; there may be a further pressure on $M$ resulting from forces impressed on the fluid in $MQ$, as in the case of gravity.

8. We may consider the pressure $Q$ as supplied by the sides of the vessel, of which the piston forms a part; $P$ and $P'$ will then represent the pressures produced on $N$ and $N'$, by the resistance of the element $M$ of the coats of the vessel: and since the pressures similarly produced by any other element, are in the same ratio, it follows that the equation is true, if they be taken to represent the whole pressure resulting from the sum of the resistances, or the whole reaction of the sides of the vessel.

$N$ and $N'$ may be taken anywhere in the fluid, and may, therefore, be supposed to form a part of the containing vessel. One or both of them may, in fact, be considered as terminating pistons, acted on by the forces $P$ and $P'$: in which last case, the relation $\frac{P}{P'} = \frac{N}{N'}$ will be necessary to the equilibrium. The proof extends to any number of surfaces, $N$, $N'$, $N''$.

9. The surface $N'$ may be of any form, of any magnitude, and in any position, and may, therefore, be taken to represent the whole interior surface of the vessel. The portion $N$, which may be considered as the termination of a piston, being excepted. Now $P' = \frac{N'.P}{N}$; hence, therefore, (*cæteris datis*) the pressure on the coats of the vessel is the greatest when the surface $N$ of the piston in contact with the fluid is the least: and, the piston, the pressure upon it, and the volume of fluid being given, the pressure on the coats of the vessel is least, when the containing surface $N'$ is the least that will contain that fluid; that is, when it is a sphere (Garnier, *Cal. Integ.* 616.). Hence, therefore, it appears that the spherical form is that best adapted to vessels containing a fluid which is subjected to pressure.

Prob. If an inverted cone contain a fluid, and a *given* pressure be applied by means of a piston to the whole of its horizontal surface; the pressure sustained by the sides of the cone is the same to whatever height it be filled.

For, if $a = \frac{1}{2}$ the angle at the vertex,

$$\frac{P'}{P} = \frac{\text{surface}}{\text{base}}$$

$$= \text{cosec}\ a;$$

$$\therefore\ P' = P\ .\ \text{cosec}\ a = \text{constant.}$$

Of cylindrical vessels containing a given quantity of fluid, the pressure produced by a given force acting on a given surface, is the least in that, whose height is equal to the diameter of its base. That being the least cylindrical surface of a given capacity.

10. It is manifest, that the general proposition we have stated is equivalent to this, that force impressed on any portion of the containing surface of a fluid, is propagated to every other equal portion of it. For, since $P$ is the pressure on the whole surface $N$, $\frac{P}{N}$ is that on each unit of it: and similarly $\frac{P'}{N'}$ is the pressure on each unit of the surface $N'$. Now

$$\frac{P}{N} = \frac{P'}{N'};$$

therefore the pressure on every unit of $N$ is propagated to every unit of $N'$: and hence it follows, that the pressure on any surface $A$ of $N$, produces an equal pressure on an equal surface $A$ of $N'$.

11. Let $M$ be a body entirely contained in the fluid. (Fig. 1.) Now the pressure produced by $P$ being referred to an unit of surface, is represented by $\frac{P}{N}$. Let $s$ and $s'$ be elementary portions of the surface of $M$, having a common projection on the plane $xy$: and let $\sigma$ and $\sigma'$ be the angles which the normals or the complements of the angles which the tangent planes to these elementary surfaces make with the same plane.

Then are $\frac{P}{N} s \sin \sigma$, and $\frac{P}{N} s' \sin \sigma'$, the pressures on $s$ and $s'$, resolved perpendicular to the plane $zy$. But $s \sin \sigma$ and $s' \sin \sigma'$ represent each the common projection of the surfaces $s$ and $s'$. The expressions $\frac{P}{N} s \sin \sigma$, and $\frac{P}{N} s' \sin \sigma'$, are, therefore, equal to one another. And the opposite pressures on any two elements of the surface, having a common projection on the plane $zy$, are equal, and therefore destroy. The body $N$ can, therefore, have no motion, whether of rotation or translation, as it respects that plane: and similar reasoning applies to the other co-ordinate planes. It follows, therefore, that if a body be wholly contained in a fluid, a pressure communicated to that fluid has no tendency whatever to alter the position of the body, provided the fluid itself remains at rest.

12. To determine the pressure on any portion of the sides of the vessel, (Fig. 1.) tending to cause a motion of translation from the plane $zy$.

Let $dz\ dy$ represent the projection of $(s)$ on the plane $zy$.

$$\therefore\ s \sin \sigma = dz\ dy;$$

$$\therefore\ \frac{P}{N} s = \frac{P}{N} \frac{dz\ dy}{\sin \sigma};$$

$$\therefore\ \text{pressure} = \frac{P}{N} \iint \frac{dz\ dy}{\sin \sigma}$$

$$= \frac{P}{N} \iint \frac{dz\ dy}{\sqrt{\left(\frac{dz}{dx}\right)^2 + \left(\frac{dz}{dy}\right)^2 + 1}}.$$

All that has been hitherto said with regard to incompressible fluids, applies, without restriction, to elastic fluids, the piston being supposed to have attained a state of repose. It is only in the manner in which this state is attained, that the difference consists.

# CHAP. II.

## ON DENSITY AND SPECIFIC GRAVITY.

13. DENSITY is the quantity of matter contained in a given volume, which volume is commonly taken to be an unit of the whole; so that in an homogeneous body the whole quantity of matter is equal to the volume multiplied into the density. Thus, calling $V$ the volume of a body, and $D$ its density, the quantity of matter it contains is represented by $V.D$.

Having no positive conception of that which is *not* matter, we can arrive at no definite idea of the *precise quantity* of that which *is*. Whilst, however, it is thus impossible to attain to an accurate knowledge of the density of any material body, we may readily institute a comparison between the densities of different bodies, and thus transfer such properties (dependant thereon) as may be predicated of one to all the rest.

That body will be best calculated to form the standard of this comparison, which is of most common occurrence; and between which and the rest the comparison is most readily instituted.

Water possesses these properties in a remarkable degree.

The density of water, as compared with that of any other substance, is the *specific gravity* of that substance. Thus, calling $D$ the density of any substance, and $S$ its specific gravity: also, representing the density of water by $D_1$, we have

$$\frac{D}{D_1} = S.$$

Similarly, if $D'$ and $S'$ be the density and specific gravity of any other substance,

$$\frac{D'}{D_1} = S';$$

$$\therefore \quad \frac{D}{D'} = \frac{S}{S'}:$$

the densities of bodies are, therefore, as their specific gravities.

Since $$\frac{D}{D_1} = S;$$

$$\therefore \quad D = S \cdot D_1:$$

$$\text{and} \quad V \cdot D = S \cdot V \cdot D_1.$$

Now, $VD$ and $VD_1$ are respectively the quantities of matter contained in the body, and in an equal volume of water. The specific gravity of a body may, therefore, be defined to be the number of times the same volume of water must be taken, to contain the same quantity of matter with it, or *the ratio of the weights of equal volumes of the body and of water.* And where $S \cdot V$ is made (as is sometimes the case) to represent the quantity of matter, it is always to be understood that it does not *strictly* represent that quantity, but the *volume* of water which contains the same quantity of matter with it.

14. If two bodies, whose densities are $D$ and $D'$, and volumes $V$ and $V'$ be mingled together; and the volume of the whole *be* the sum of the volumes of the parts; then, calling $D''$ the density of the compound.

Since the *whole* quantity of matter it contains, is equal to the sum of the quantities of matter in the parts,

$$D''(V + V') = V.D + V' \cdot D':$$

or $D_1$ representing, as before, the density of water,

$$\frac{D''}{D_1}(V+V') = V.\frac{D}{D_1} + V'.\frac{D'}{D_1};$$

$$\therefore\quad S''(V+V') = V.S + V'.S';$$

$$\therefore\quad S'' = \frac{V.S + V'.S'}{V + V'}.$$

$S''$, $S$ and $S'$ representing respectively the specific gravities of the compound and its parts.

In all chemical combinations, the volume of the compound is found to be other than the sum of the volumes of the component parts. The above theorem applies, therefore, only to the case of mechanical admixture.

## CHAP. III.

### ON THE EQUILIBRIUM OF FLUIDS ACTED UPON BY GIVEN ACCELERATING FORCES.

The property of the uniform distribution of pressure, applied at the surface of fluids, belongs to their *nature*, and is common to all of them. Fluids, however, whose particles are acted upon by gravity, or other accelerating forces, exert, besides the pressure thus propagated from their surfaces, and uniformly diffused throughout them, a further pressure, dependant upon their density and the magnitude of the forces impressed, and variable from one point in them to another.

15. To estimate the quantity of this variable force at any point in the fluid, it is conceived to be applied uniformly to an unit of surface. Throughout the following pages, the pressure thus referred to an unit of surface is represented by $p$; and it must clearly be understood, that this symbol

is not taken to represent any pressure actually produced by the fluid, but that which would be produced if the pressure on the point (or rather an element at the point) under consideration, were uniformly applied to an unit.

16. Let the irregular vessel $MN$ (Fig. 3.) contain a fluid acted upon by the constant force of gravity. Suppose the whole to be cut horizontally by the plane $KL$, and, the fluid being at rest, let its upper portion $ML$ become solid excepting only the vertical tube $PQ$ whose section is $(k)$. Then will the circumstances of the equilibrium and the pressure, on every point of the fluid, remain precisely as before, nothing having been added or taken away from the forces impressed.

Now it is evident that the fluid in $PQ$ is acted on by a moving force in the direction of gravity equal to its weight; also since the sides of the tube are vertical and nothing is opposed to this force in the direction of its action, it is wholly effective, and exerts its whole pressure on its lowest section. Now by (Art. 6.) this pressure will generate an equal pressure on an equal surface any where taken in the fluid $KN$. Also since the fluid in $KN$ does not otherwise press upon the plane $KL$ than as it communicates the pressure of $PQ$, (its gravity impelling it from that plane), it follows that this pressure is the only force effective on $KL$, and on the fluid surface in contact with it, and, therefore, generally, that the pressure on any portion of a horizontal section, is equal to the weight of a vertical column of the fluid, whose base is of the same area with it, and which reaches to the surface.

Let $mn$ be any other surface, $KL$ and $kl$ horizontal sections through its highest and lowest points. Then is the pressure on $mn$, manifestly greater than that on an equal portion of $KL$: and less than that on an equal portion of $kl$; and, therefore, the pressure upon it differs from the pressure on an equal portion of $KL$, by less than the weight of the column $Qq$. If, therefore, the plane $mn$ be taken so small that $Qq$ may vanish when compared with $PQ$, the pressure upon it will be accurately represented by the weight of $PQ$,

and, this is true whatever be the form or position of $mn$, the base of $PQ$ being as before of the same area.

Refer the pressure on $mn$ to an unit of surface; let $z$ be the depth, $D$ the density of the fluid, and $G$ the force of gravity for any value of $z$, then will the weight of the column $PQ$, when referred to an unit, be $\int DGdz$, and therefore

$$p = \int DGdz \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (\beta).$$

17. It has been shewn that if the plane $KL$ be horizontal the pressure upon every equal area of it is the same, and further, that it is the weight of *any* vertical column whose base is equal to that area, and which reaches to the surface of the fluid. Hence, therefore, it appeared that the weights, and, therefore, the heights of all such vertical columns are the same. Since, then, the surface of the fluid is every where at the same height above the horizontal plane $KL$, it is itself horizontal.

18. Similarly, if there be *two* fluids of different densities in the same vessel, their common surface is horizontal. For taking a horizontal plane in the lower fluid, as before, the pressure on every equal area of it is the same, and is equal to the weight of any vertical column extending from the plane to the surface. Hence, the weight of all such vertical columns is the same, and the upper surface being horizontal, their height is the same, they must, therefore, all contain the equal portions of the two fluids, and the heights of the lower columns, that is, the distances of the different points in their common surface from horizontal plane must all be the same, or their common surface itself must be horizontal.

And, in the same manner, if there be any number of different fluids contained in the same vessel, taking horizontal planes in any two adjacent fluids, it appears that since the pressures on equal surfaces, throughout these planes are the same, and equal to the weight of any of the superincumbent columns (taking the equal columns incumbent on the higher plane from those on the lower), the weights of all the columns between the two planes are the same, and their heights are manifestly the same since both planes are horizontal; therefore

they contain all the same quantities of the two fluids, and the distance of every point of the common surface of the fluids, from either plane, is the same.

These results follow at once from the equation $(\beta)$; from whence it appears, that in every *possible* case of pressure, $DGdz$ is an exact differential, and, therefore, $D$ constant, or a function of $z$. When, therefore, $z$ is constant, $D$ is constant; or all horizontal planes are of uniform density. And, when $p$ is constant, (as at the surfaces of fluids) $z$ is constant; or the surfaces of fluids are horizontal.

19. From the above it appears, that the common surface of the atmosphere and any fluid on the earth's surface, is horizontal: and further, (the air being of variable density) that the different layers or strata of air in different states of density, are disposed horizontally, and parallel to each other.

*On the Equilibrium of a Fluid in a System of communicating Vessels.*

20. It has been shewn to be a condition necessary to the equilibrium of a fluid acted upon by gravity, that, being intersected by a horizontal plane, the pressure on every portion of that plane should be the same. And the demonstration of this proposition obtains, whatever be the form of the containing vessel, provided only the parts of it communicate, so that the fluid may be, in any direction, continuous.

Suppose two vessels to communicate by means of a tube or common aperture, and let a fluid be poured into one of them. When the whole is in equilibrium, let it be intersected by a horizontal plane; which is, therefore, one of equal pressure. Now, the pressure on every unit of this plane is the weight of a superincumbent column of the fluid, together with a superincumbent column of the atmosphere above it: and taking the atmospheric pressure to be the same over every unit of the surface in both vessels, it follows, that the weights of the superincumbent columns of the fluid, or their

heights, are the same. The surface, therefore, of the fluid is at the same distance from the same horizontal plane or at the same level, in both vessels.

21. If the pressure of the atmosphere, or any part of it, be removed from the surface of the fluid in one of the vessels, remaining as before, over the surface of that in the other, the equality of pressure on the horizontal plane we have taken, will be destroyed; and the conditions of equilibrium no longer obtaining, that surface which sustains the less pressure will ascend and the other descend, until the equilibrium is restored by, the lesser pressure of the atmosphere on an unit of the former surface, added to the weight of the increased column of fluid; equalling the greater pressure on an unit of the other surface, added to the weight of the diminished column of fluid.

Thus, (Fig. 25.) if $p$ represent the pressure of the atmosphere on an unit of the surface $A$, and $p'$ on an unit of $A'$: and $BB'$ be any horizontal plane. When the fluid is at rest,

$$p + \text{weight } AB = p' + \text{weight } A' \; B';$$

$$\therefore \quad p = p' + \text{weight } A'A''.$$

If the pressure of the atmosphere be wholly removed from the surface $A'$, that is, if $p' = 0$,

$$p = \text{weight } A'A''.$$

The surface of the fluid in one vessel may thus be raised above that in the other, until the weight of the column raised is equal to the pressure of the atmosphere on a portion of the other surface, equal to its base.

22. If the fluid in $AB$ be not of the same density with that in $A'B'$, it is clear, that in order to preserve the equality of pressure on the plane $BB'$, we must have

$$\overline{AB} \times D = \overline{A'B'} \times D',$$

or $$\overline{AB} = \overline{A'B'} \cdot \sigma:$$

where $D$ and $D'$ are the densities of the two fluids, and $\sigma$ the ratio of their specific gravities.

Ex. 1. Given quantities of different fluids are contained in a circular tube; it is required to determine their position of equilibrium when at rest.

Let $EQ$ (Fig. 16.) be the heavier fluid, and $FQ$ the lighter.

$$\text{rad} = 1 \qquad \angle ECQ = \alpha \qquad \angle FCQ = \beta;$$

$$\therefore\ ME' = \overline{MF'} \cdot \sigma.$$

Let $\qquad AP = AQ = \theta;$

$$\therefore\ ME' = \cos\theta - \cos(\alpha - \theta)$$

$$MF' = \cos\theta - \cos(\beta + \theta);$$

$$\therefore\ \{\cos\theta - \cos\alpha\ \cos\theta - \sin\alpha \cdot \sin\theta\}$$

$$= \{\cos\theta - \cos\beta\ \cos\theta + \sin\beta\ \sin\theta\}\ \sigma;$$

$$\therefore\ \{1 - \cos\alpha - \sin\alpha \cdot \tan\theta\} = \{1 - \cos\beta + \sin\beta\ \tan\theta\}\ \sigma;$$

$$\therefore\ \tan\theta = \frac{(1 - \cos\alpha) - (1 - \cos\beta)\ \sigma}{\sin\alpha + \sigma \cdot \sin\beta}.$$

Ex. 2. Equal quantities of two fluids, the ratio of whose specific gravities is represented by $\sigma$, are contained in a cycloidal tube. To determine their positions of equilibrium. Let $EP$ (Fig. 7.) be the heavier fluid, and $FP$ the lighter.

$$BF' = x \qquad BE' = x_{\prime} \qquad BP' = x_{\prime\prime},$$

length of each arc $PE$ and $PF = \sqrt{2al}$;

$$\therefore\ 2\sqrt{2ax} + 2\sqrt{2ax_{\prime}} = 2\sqrt{2al},$$

$$\text{or}\quad x^{\frac{1}{2}} + x_{\prime}^{\frac{1}{2}} = l^{\frac{1}{2}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Now, $\qquad P'E' = P'F' \cdot \sigma,$

$$\text{or}\quad x - x_{\prime\prime} = (x_{\prime} - x_{\prime\prime})\ \sigma;$$

$$\therefore\ x_{\prime\prime} = \frac{x - x_{\prime}\sigma}{1 - \sigma}.$$

Also, since $$PF = PE;$$

$$\therefore\ 2\sqrt{2ax} - 2\sqrt{2ax_{\backslash\backslash}} = 2\sqrt{2ax_{\backslash}} + 2\sqrt{2ax_{\backslash\backslash}}:$$

or
$$x^{\frac{1}{2}} - x_{\backslash}^{\frac{1}{2}} = 2x_{\backslash\backslash}^{\frac{1}{2}}$$
$$= 2\left(\frac{x - x_{\backslash}\sigma}{1-\sigma}\right)^{\frac{1}{2}} \ldots\ldots\ldots (2)$$

eliminating between the equations (1) and (2), we obtain

$$x = \frac{1}{16}\left(\frac{1+3\sigma}{1+\sigma}\right)^2 . l \quad x_{\backslash} = \frac{1}{16}\left(\frac{3+\sigma}{1+\sigma}\right)^2 . l.$$

## CHAP. IV.

### ON THE PRESSURE SUSTAINED BY THE SURFACES OF VESSELS CONTAINING FLUIDS, OR IMMERSED IN THEM.

23. Since, if an aperture be any where made in the sides of a vessel containing fluid, acted on by gravity, the fluid will escape; it is clear that the surface of the vessel every where sustains a certain pressure.

24. A *surface* can sustain no pressure, except in the direction of its normal.

25. The internal pressure on the coats of a vessel, when *filled* to a certain depth with a fluid acted on by gravity, is the same with the external pressure upon it when *immersed* in the same fluid to the same depth. For the pressure on an element, in either case, is equal to the weight of a vertical column, whose base is of the same area with the element, and which extends to the surface of the fluid. Now this column is manifestly the same in either case. And this being true for every element of the surface, is true for the whole.

26. Hence it appears, that if a vessel * be wholly immersed, and filled with the fluid in which it is immersed, the pressures (severally and in the whole) destroy, and have no tendency to alter its form. And thus, although the actual pressure of a fluid (as, for instance, that of the atmosphere) be considerable, it has no tendency whatever to *alter the form* of a vessel, however fragile, which is *wholly* immersed and filled with it.

27. Hence, also, if a vessel containing one fluid be immersed in another, the pressure sustained by that part of its sides which is in contact with the contained fluid, and tends to disturb its form, is equal to the difference between the actual pressure of this fluid, and that which would have been sustained, had the vessel been filled, to the same depth, with the fluid in which it is immersed.

28. The pressure, therefore, of fluids on the sides of the vessels which contain them, is not materially affected by the immersion of the vessel and fluid in the atmosphere. For the actual pressure differs from that in vacuo, by the pressure† that would be produced by an equal quantity of air contained in the vessel. And in all cases where the former pressure is appreciable, the latter may be neglected, as compared with it.

29. Since the pressure on any element of the sides of a vessel containing fluid, is equal to the weight of a vertical column of the same fluid of the same depth, and whose horizontal section is of the same area, it appears that the pressure on the lower parts of it is greater than that on the higher, in the proportion of their depth.

Since the *quantity* of any material is a principal element of its strength, it is clear that the lower parts of vessels should be of greater thickness than the higher.

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* No account is here taken of the thickness of the vessel.

† The pressure here meant is that which arises simply from the *weight* of the air considered as a *liquid,* and is independent of its elasticity.

If the strength of the material be taken to vary directly as its quantity, and any portion of the containing surface of a vessel be a plane, of which $AB$ (Fig. 5.) is a vertical section, then $AA'$ being the surface of the fluid, and $P$ any point in $AB$, draw $PN$ perpendicular to it, and let $PN$ be the thickness of the containing substance just necessary to sustain the pressure on $P$. Draw $AC$ through $N$; then $CAB$ will be a vertical section of the vessel, when its thickness is precisely that which is requisite to support the fluid it contains. For taking any other point $P'$, and drawing $P'M'$ and $P'N'$ respectively parallel to $PM$ and $PN$, we have

$$\frac{\text{pressure at } P}{\text{pressure at } P'} = \frac{PM}{P'M'}$$

$$\frac{\text{strength at } P}{\text{strength at } P'} = \frac{PN}{P'N'};$$

$$\text{but} \quad \frac{PM}{P'M'} = \frac{PN}{P'N'};$$

$$\therefore \frac{\text{pressure at } P}{\text{pressure at } P'} = \frac{\text{strength at } P}{\text{strength at } P'},$$

$$\text{but} \quad \text{pressure at } P = \text{strength at } P;$$

$$\therefore \text{pressure at } P' = \text{strength at } P'.$$

30. Having given the interior surface of a vessel of fluid, to find what must be its exterior surface that the pressure on every point of it may be proportional to its strength, this last being taken to vary as the quantity of material in the direction of the pressure.

Suppose the interior surface, one of revolution about a vertical axis, and let $BAQP$ (Fig. 6.) be a section of the vessel made through this axis. Take $PQ$ a normal to the interior surface at $P$; draw $PM$ and $QN$ perpendiculars to the surface $OA$ of the fluid.

Let $\quad OM = x, \qquad PM = y,$

$\qquad ON = X, \qquad QN = Y.$

Then, since the pressure is in the direction of $PQ$, and varies as $PM$, and that the strength is by hypothesis, as $PQ$,

$$PQ \propto y = my;$$

$$\therefore (X-x)^2 + (Y-y)^2 = m^2y^2 \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Also, $$(X-x) = (y-Y) \tan PQN$$
$$= (y-Y) \tan MPK$$
$$= (y-Y) \frac{dy}{dx} \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Now, since the curve $BP$ is given, we have

$$fxy = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

And eliminating $x$ and $y$ between these three equations, we obtain an equation in $X$ and $Y$, to the curve by the revolution of which about the axis of the interior surface the exterior is described.

Since the perimeter of every horizontal section of a surface of revolution varies as the corresponding ordinate of the generating curve; since also the pressure upon every point of it varies as the abscissa, it follows that the pressure on every such section will be the same, if the product of the ordinate and abscissa of the generating curve be a constant quantity. A known property of the rectangular hyperbola—an asymptote being taken for the axis of the abscissæ.

31. Let $\Delta S$ be an element of the surface of a vessel, $p$ the pressure on any point in it referred to an unit of surface. Then, if we consider the pressure to be the *same* for every point in $\Delta S$, it will be represented by $p \cdot \Delta S$, and that on the whole surface by $\Sigma p \cdot \Delta S$. But the pressure *varies* from one point of the vessel to another, however near they may be to each other; the pressure is, therefore, *not the same* for every point in $\Delta S$, and $\Sigma p \cdot \Delta S$ does not represent the true pressure on the sides of the vessel, but continually approaches it as $\Delta S$ diminishes, attaining it for no finite value whatever of that quantity. The true pressure, therefore, is represented by

$$\int p\, dS \ldots\ldots\ldots\ldots\ldots\ldots (\gamma).$$

If the fluid be incompressible, and the force be that of gravity, considered as constant,

$$p = \int Dgdz = Dgz,$$

$z$ being measured from the surface;

$$\therefore \text{ pressure} = Dg\int zdS.$$

Now, if $z_1$ be the distance of the center of gravity of the surface, or any portion of the surface of the vessel, from the horizontal surface of the fluid, we have

$$S \,.\, z_1 = \int zdS;$$

$$\therefore \text{ pressure} = D \,.\, S \,.\, gz_1 \ldots\ldots\ldots\ldots\ldots (\delta).$$

32. Hence, therefore, the pressure on the surface, or any portion of the surface of a vessel containing fluid, is equal to its weight, supposing it of the same density with the fluid, multiplied by the perpendicular depth of its center of gravity below the horizontal surface of the fluid.

33. If, in addition to the weight of the contained fluid, a force $Pg$ be made to act upon it by means of a piston pressing on a surface $S'$, the pressure on the remainder becomes

$$D \,.\, S \,.\, z_1 . g + \frac{SPg}{S'} = Sg\left(z_1 \,.\, D + \frac{P}{S'}\right).$$

34. If the portion of the vessel on which the pressure is required to be determined, be an horizontal plane, as, for instance, its base, $z_1$ will represent its depth. And the expression $DSgz_1$ will be the weight of a vertical cylinder or prism of fluid continued from its base to its surface. Now this is true whatever be the form of the other part of the vessel. Vessels, therefore, of whatever form, when filled to the same height with fluid, exert the same pressure on their bases, provided those bases be planes in a horizontal position, and of the same magnitude. And this is true whatever be the positions of the bases, provided the vessels be filled to the same height ($z_1$) above their centers of gravity.*

* That is, the centers of gravity of their bases.

35. It is manifest, that if a vessel be wholly immersed in a fluid, and made to revolve about its center of gravity, the pressure on its surface, whether internally or externally, will remain unaltered, all the terms of the expression $D . S . g z_1$ continuing the same.

36. Of curves, whose length and the positions of whose points of suspension are given, the center of gravity of the catenary is the lowest; and therefore the pressure upon it, when filled with fluid, is the greatest.

37. If the surface be referred to three co-ordinate planes, generally,

$$dS = \frac{dx\,dy}{\sqrt{\left(\frac{dz}{dx}\right)^2 + \left(\frac{dz}{dy}\right)^2 + 1}};$$

$$\therefore \text{ pressure} = \iint \frac{p\,dx\,dy}{\sqrt{\left(\frac{dz}{dx}\right)^2 + \left(\frac{dz}{dy}\right)^2 + 1}}:$$

in the case of gravity,

$$\text{pressure} = g \iint \frac{z\,dx\,dy}{\sqrt{\left(\frac{dz}{dx}\right)^2 + \left(\frac{dz}{dy}\right)^2 + 1}}.$$

*On the Center of Pressure.*

38. If a plane surface sustain the pressure of a fluid, forming part of a vessel which contains, or is immersed in it, or otherwise; since the various pressures on different points of the plane are perpendicular to it, they are parallel to one another. The point where the resultant of these parallel forces cuts the plane, is called its *center of pressure*, and is manifestly that point to which, if a pressure equal to the whole pressure on the plane be applied in an opposite direction, it will sustain it, and hold the plane in equilibrium.

Conceive the plane (Fig. 8.) to be produced to the surface of the fluid, and take its intersection with that surface for

the axis of ($y$), and a line perpendicular to this drawn in the plane itself for the axis of $x$.

Now, generally, if $x_1$ and $y_1$ be co-ordinates of the point where the resultant of any number of parallel forces applied to a plane perpendicular to their common direction, intersects it, and $x$, $y$ be co-ordinates of the point of intersection of any one of the forces ($p$) with the plane, we have

$$x_1 = \frac{\Sigma p x}{\Sigma p}$$

$$y_1 = \frac{\Sigma p y}{\Sigma p}.$$

In the case in question, the force $p$, or the pressure on any element of the plane is the weight of a column of fluid of the same depth, and whose base is equal in area to the element. The pressure ($p$) on the element $dxdy$ is, therefore, $Dgx \sin\theta dxdy$, $\theta$ being the inclination of the plane to the surface of the fluid, and therefore $x\sin\theta$ the depth of the point $x$, $y$. Hence, therefore, substituting for ($p$) its value, we have

$$\left.\begin{aligned} x_1 &= \frac{\iint Dgx^2 \sin\theta \,.\, dx \,.\, dy}{\iint Dgx \sin\theta \,.\, dx \,.\, dy} = \frac{\iint Dx^2 dxdy}{\iint Dxdxdy} \\ y_1 &= \frac{\iint Dgxy \sin\theta dxdy}{\iint Dgx \sin\theta dxdy} = \frac{\iint Dxydxdy}{\iint Dxdxdy} \end{aligned}\right\} \ldots\ldots(\epsilon).$$

If the density be constant, these resolve themselves into

$$\left.\begin{aligned} x_1 &= \frac{\iint x^2 dydx}{\iint xdxdy} = \frac{\int x^2 ydx}{\int xydx} \\ y_1 &= \frac{\iint xydydx}{\iint xdxdy} = \tfrac{1}{2}\,.\,\frac{\int xy^2 dx}{\int xydx} \end{aligned}\right\} \ldots\ldots\ldots\ldots(\zeta).$$

39. The former of these expressions coincides with that determining the distance of the center of oscillation or percussion of the plane from the axis of ($y$), supposing it to revolve about that axis. Hence, therefore, it appears, that if the plane be symmetrical about the axis of $x$, its center of pressure and its centers of oscillation and percussion, when made to revolve about the axis of $y$, coincide.

40. If we measure $x$ from a point $E$, distant by $EO$ $(=k)$, from the intersection of the axis of $x$ with the surface, the equation becomes

$$x_1 = \frac{\int (x^2 + kx)\, y dx}{\int (x + k)\, y dx}$$

$$y_1 = \tfrac{1}{2} \frac{\int (x + k)\, y^2 dx}{\int (x + k)\, y dx}.$$

In the case in which $AD$ and $BC$ converge towards $CD$, $k$ must be taken to represent the distance $OG$, so that $OM$ will be represented by $k - x$; and we shall have for the general expressions

$$x_1 = \frac{\int (kx \pm x^2)\, y dx}{\int (x \pm k)\, y dx} \ldots (1), \quad y_1 = \tfrac{1}{2} \frac{\int (k \pm x)\, y^2 dx}{\int (k \pm x)\, y dx} \ldots (2).$$

41. The quantity $\theta$ is eliminated from all the above expressions; whence it appears that, $k$ remaining the same, the position of the center of pressure on the plane is not affected by a variation in its inclination to the horizon, and will, therefore, remain the same, if the plane be made to revolve through any angle about $O$ as an axis.

42. If we make $k =$ infinity,

$$x_1 = \frac{\int xy dx}{\int y dx} \qquad y_1 = \tfrac{1}{2} \frac{\int y^2 dx}{\int y dx},$$

which are the known expressions determining the position of the center of gravity. Hence, therefore, it appears, that as the plane is sunk deeper in the fluid, its center of pressure continually approaches its center of gravity; which point, however, it never accurately coincides with.

The center of pressure lies below or above the center of gravity, according as $\dfrac{\int x^2 y dx + k \int xy dx}{\int xy dx + k \int y dx} >$ or $< \dfrac{\int xy dx}{\int y dx}$; that is, according as $(\int x^2 y dx)\,(\int y dx) >$ or $< (\int xy dx)^2$: which expression being independent of $k$, it appears, that if the centre of pressure be in any one position below the centre of gravity, it will in every other.

Ex. 1. Suppose the plane a trapezoid, of which the parallel sides $AB$ and $CD$ (Fig. 9.) are horizontal, produce the plane to meet the surface of the fluid in $Oy$, and the sides $DA$ and $CB$ to intersect one another in $F$. Draw $OFx$ perpendicular to $AB$ and $CD$; then is it also perpendicular to $Oy$, since $Oy$ and $BA$ are both horizontal and in the same plane, and therefore parallel. Draw $PQ$ parallel to $Oy$, and let it be represented by $y$: let $EM=x$, $OE=k$, $AB=a$, $CD=a'$, $EG=b$;

$$\therefore \text{ by } \triangle^s, \ \frac{FE+x}{FE}=\frac{y}{a}; \qquad \therefore \ \frac{x}{FE}=\frac{y}{a}-1$$

$$\frac{FE+b}{FE}=\frac{a'}{a}; \qquad \therefore \ \frac{b}{FE}=\frac{a'}{a}-1.$$

Hence,
$$\frac{x}{b}=\frac{\dfrac{y}{a}-1}{\dfrac{a'}{a}-1};$$

$$\therefore \ \frac{y}{a}=\frac{x}{b}\left(\frac{a'}{a}-1\right)+1,$$

$$y=\frac{x(a'-a)+ab}{b}.$$

Substituting, therefore, for $y$, its value in the equation (1),

$$x_1=\frac{\int(x^2+kx)\{ab+x(a'-a)\}\,dx}{\int(x+k)\{ab+x(a'-a)\}\,dx},$$

and taking each integral, from $x=0$ to $x=b$,

$$x_1=\frac{ab(\frac{1}{3}b^3+\frac{1}{2}kb^2)+(a'-a)(\frac{1}{4}b^4+\frac{1}{3}b^3k)}{ab(\frac{1}{2}b^2+kb)+(a'-a)(\frac{1}{3}b^3+\frac{1}{2}b^2k)}$$

$$x_1=\frac{a(4b^2+6kb)+(a'-a)(3b^2+4bk)}{2a(3b+6k)+(a'-a)(4b+6k)}$$

$$=\frac{b^2(3a'+a)+bk(4a'+2a)}{b(4a'+2a)+k(6a'+6a)}$$

$$=\frac{1}{2}b\,\frac{b(3a'+a)+2k(2a'+a)}{b(2a'+a)+3k(a'+a)}.$$

When $k = 0$, or one side of the plane coincides with the surface, we have

$$x_1 = \tfrac{1}{2} b \cdot \frac{3a' + a}{2a' + a}.$$

If $a$ or $a'$ equal 0, the trapezoid resolves itself into a triangle, with its vertex in the one case upward, and in the other downward, and we have

$$x_1 = \tfrac{1}{2} b \cdot \frac{3b + 4k}{2b + 3k}, \text{ or } x_1 = \tfrac{1}{2} b \cdot \frac{2k - b}{3k - b}:$$

when $k = 0$, these become

$$x_1 = \tfrac{3}{4} b, \text{ or } x_1 = \tfrac{1}{2} b.$$

If $a_1 = a$, or the trapezoid become a parallelogram,

$$x_1 = \tfrac{1}{3} b \cdot \frac{2b + 3k}{b + 2k}: \text{ or when } k = 0,\ x_1 = \tfrac{2}{3} b.$$

When $k = \infty^{ty}$,

$$x_1 = \tfrac{1}{3} b \, \frac{2a' + a}{a' + a},$$

which is the known expression for the distance of the center of gravity from the side $AB$ of the trapezium. It appears, therefore, that as the plane descends, the center of pressure continually approaches the center of gravity. Also, since the whole distance through which its position varies, as the side $AB$ descends from the surface of the fluid, is represented by

$$\tfrac{1}{2} b \, \frac{3a' + a}{2a' + a} - \tfrac{1}{3} b \, \frac{2a' + a}{a' + a},$$

$$\text{or} \qquad \tfrac{1}{6} b \cdot \frac{a'^2 + 4a a' + a^2}{(a' + a)(2a' + a)},$$

which is a positive quantity; it follows, that the center of pressure is always below the center of gravity, to whatever depth the plane be sunk, and lies between that point and the point determined by the equation

$$x_1 = \tfrac{1}{2} b \cdot \frac{3a' + a}{2a' + a}.$$

It remains now to determine the distance of the center of pressure from the axis $Ox$. Let $MP = my$, $MQ = ny$. Then, recurring to equations ($\epsilon$), we have

$$y_1 = \frac{\iint xy\,dx\,dy}{\iint x\,dx\,dy}.$$

The first integration, performed with respect to $y$, must be taken from $y = -my$ to $y = ny$. Hence

$$y_1 = \frac{n^2 - m^2}{2(n+m)} \frac{\int xy^2 dx}{\int xy\,dx} = \frac{n-m}{2} \frac{\int xy^2 dx}{\int xy\,dx}.$$

Substituting for $y$, and integrating from $x = 0$ to $x = b$,

$$y_1 = \frac{n-m}{4} \cdot \frac{b(3a'^2 + 2aa' + a^2) + 4k(a'^2 + aa' + a^2)}{b(a + 2a') + 3k(a + a')}.$$

Let $a = 0$, or let the figure become a triangle with its vertex upwards;

$$\therefore\ y_1 = \frac{n-m}{4} \cdot \frac{3b + 4k}{2b + 3k} \cdot a'.$$

Let $a' = 0$, or let the figure be a triangle with its vertex downwards. Then, changing the sign of $b$,

$$y_1 = \frac{n-m}{4} \cdot \frac{4k - b}{3k - b} \cdot a.$$

In either case of the triangles, the motion of the center of pressure, as the figure descends, is in a right line, passing through the vertex, and inclined to the perpendicular $EG$ at an angle whose tangent is $\frac{y_1}{x_1}$ or $\frac{n-m}{2} \cdot \frac{a'}{b}$ in the case of the vertex upwards, and $\frac{y_1}{b - x_1}$ or $\frac{n-m}{2} \cdot \frac{a}{b}$ in the case of the vertex downwards. It consequently bisects $DC$.

Let it be required to determine where a *single* hoop must be fixed to hold together the staves of a barrel filled with fluid. The staves being supposed to be similar, and each to present a plane surface to the fluid, and to revolve

freely about the hoop, it is clear that the hoop must pass through the center of pressure of each stave. Its distance from the top of the barrel will, therefore, be determined by the equation

$$x_1 = \tfrac{1}{2} b \cdot \frac{3a' + a}{2a' + a},$$

($a$) and ($a'$) being the lengths of the two ends of a stave. Or if ($\alpha$) be the ratio of the circumferences of the upper and lower ends of the cask respectively, since $\frac{a}{a'} = \alpha$;

$$\therefore\ x_1 = \tfrac{1}{2} b \cdot \frac{3 + \alpha}{2 + \alpha}.$$

If the vessel be an upright cone, $\alpha = 0$, and $x_1 = \frac{3}{4} b$. If it be an inverted cone, $\alpha = \infty^{\text{ty}}$, and $x_1 = \frac{1}{2} b$.

If the staves be prevented revolving inwards by the bottom of the cask, the hoop will be best placed when nearest the line of pressure, but must not be above it.

Ex. 2. Suppose the plane a parabola. Here, taking the origin at the vertex, $y = c\sqrt{x}$. Substituting, therefore, in equation (1), we have

$$x_1 = \frac{\int (x^2 + kx)\sqrt{x}\, dx}{\int (x + k)\sqrt{x}\,.\, dx}$$

$$= \frac{\int \left(x^{\frac{5}{2}} + kx^{\frac{3}{2}}\right) dx}{\int \left(x^{\frac{3}{2}} + kx^{\frac{1}{2}}\right) dx}$$

$$= \frac{\frac{2}{7} x^{\frac{7}{2}} + \frac{2}{5} k x^{\frac{5}{2}}}{\frac{2}{5} x^{\frac{5}{2}} + \frac{2}{3} k x^{\frac{3}{2}}};$$

taking the integral from 0 to $x$. Hence, dividing by $\frac{2}{5} x^{\frac{3}{2}}$,

$$x_1 = x \cdot \frac{\frac{5}{7} x + k}{x + \frac{5}{3} k}.$$

When $k = 0$, or the vertex of the parabola touches the surface of the fluid,

$$x_1 = \frac{5}{7}x.$$

When $k = \infty^{ty}$, or the parabola is plunged to an infinite depth in the fluid,

$$x_1 = \frac{3}{5}x.$$

Hence it appears, that to whatever depth the parabola be sunk, the center of pressure always lies below the center of gravity, and that its position varies between that point and a point distant from the vertex by $\frac{5}{7}^{\text{ths}}$ of the axis, as the parabola descends.

Ex. 3. To find the center of pressure of the quadrant of a circle, the diameter coinciding with the surface of the fluid,

$$x_1 = \frac{\int x^2 y\, dx}{\int y x\, dx} = \frac{\int x^2 (a^2 - x^2)^{\frac{1}{2}}\, dx}{\int x (a^2 - x^2)^{\frac{1}{2}}\, dx}.$$

Let $P = x(a^2 - x^2)^{\frac{3}{2}}$; $\therefore \dfrac{dP}{dx} = (a^2 - x^2)^{\frac{3}{2}} - 3x^2 (a^2 - x^2)^{\frac{1}{2}}$

$$= a^2 (a^2 - x^2)^{\frac{1}{2}} - 4x^2 (a^2 - x^2)^{\frac{1}{2}};$$

$$\therefore x^2 dx (a^2 - x^2)^{\frac{1}{2}} = -\frac{dP}{4} + \frac{a^2}{4}(a^2 - x^2)^{\frac{1}{2}} dx;$$

$$\therefore \int x^2 (a^2 - x^2)^{\frac{1}{2}} dx = -\frac{x(a^2 - x^2)^{\frac{3}{2}}}{4}$$

$$+ \frac{a^2}{4} \cdot \text{circular area } \{\cos(x) \text{ rad } (a)\} + C.$$

Now, $C = 0$;

$$\therefore \int x^2 (a^2 - x^2)^{\frac{1}{2}} dx = \frac{\pi a^4}{16}, \text{ when } x = a,$$

$$\text{and } \int x\, dx \sqrt{a^2 - x^2} = -\tfrac{1}{3}(a^2 - x^2)^{\frac{3}{2}} + C = \frac{a^3}{3}:$$

$$\therefore x_1 = \frac{3}{16} \cdot \pi \cdot a.$$

$$\text{Again, } y_1 = \tfrac{1}{2}\frac{\int y^2 x\,dx}{\int xy\,dx} = \frac{\int x(a^2-x^2)\,dx}{2\int x(a^2-x^2)^{\frac{1}{2}}\,dx}$$

$$= \tfrac{1}{2}\,\frac{\tfrac{1}{2}a^2x^2 - \tfrac{1}{4}x^4}{-\tfrac{2}{3}(a^2-x^2)^{\frac{3}{2}} + \dfrac{2a^3}{3}}$$

$$= \frac{\dfrac{a^4}{4}}{\dfrac{2a^3}{3}}, \text{ when } x = a,$$

$$= \tfrac{3}{8}a.$$

*On the Resolution of Fluid Pressure, and the Center of Pressure in Curved Surfaces.*

43. Let $G$ (Fig. 10.) be the center of gravity of an elementary area $K$, forming any portion of the surface of a vessel sustaining the pressure of a fluid in which it is immersed, or which it contains.

Let the area $K$ be supposed a plane. Draw $GN$ perpendicular to the surface of the vessel, and $GM$ to the surface of the fluid, and let them meet this surface in $N$ and $M$. Let $I$ be the inclination of $GN$ to the surface of the fluid: then is $\left(\frac{\pi}{2} - I\right)$ the inclination of the area $K$, which is perpendicular to $GN$. Now, the pressure of the fluid on $K$ is perpendicular to its surface, or in the direction $GN$, and equal to $D \cdot g \cdot \overline{GM} \cdot K$. And of this force, that which is effective in the direction of gravity, is $D \cdot g \cdot \overline{GM} \cdot K \sin I$. But $K \sin I = K \cos\left(\frac{\pi}{2} - I\right) = K \cdot$ cos of inclination of $K$ to surface of fluid $=$ projection $(K')$ of $K$ on the surface. Therefore the whole pressure on $K$ in the direction of gravity, is represented by $D \cdot g \cdot \overline{GM} \cdot K'$, or it is equal to the weight of an immediately superincumbent column of the fluid. And the whole pressure upon the vessel in the direction of gravity

is represented by the sum of such columns. And this is true, however small the planes ($K$), and therefore the superincumbent columns be taken, that is, however near their sum be made to approach to a volume of fluid superincumbent to the whole surface of the vessel, or extending from it to the surface of the fluid.

44. That is, in the case of a vessel containing fluid, the *vertical* pressure is equal to the weight of the fluid contained: in the case of a body immersed in fluid, it is equal to the weight of the fluid displaced.

45. Let $K''$ be the projection of $K$ on a vertical plane anywhere taken in the fluid; and let the projecting surface be continued to intersect the opposite side of the vessel in $k$, then is $K''$ the common projection of the surfaces $K$ and $k$. Now, it may be shown precisely as in the last case, that the pressures on the surfaces $K$ and $k$, perpendicular to the plane of projection, are each equal to the weight of a column of the same height with their common depth $GM$, and having for its base the common projection $K''$.

These pressures are, therefore, equal to one another; and being applied to the vessel in opposite directions, they destroy. And the same being true for all other portions of it, similarly taken, it follows, that all the horizontal pressures estimated in directions perpendicular to the plane of projection, occur (at the same perpendicular depth) in pairs, which mutually destroy: and hence that, not only have they no tendency, on the whole, to cause motion in the vessel perpendicular to this plane; but farther, that their momenta about a horizontal axis parallel to it, severally vanish. All the conditions of equilibrium, with regard to the plane of projection, are therefore fully satisfied. Now this plane is any vertical plane whatever. Motion does not, therefore, take place perpendicular to any such plane, that is, in any horizontal direction whatever. In whatever position, therefore, or to whatever depth a body be immersed, the pressure of the fluid has no tendency whatever to communicate to it a lateral motion.

46. Suppose the portion $k$ of the vessel to be removed. The horizontal pressures will then, as before, destroy, with regard to the rest of the vessel, but on $K$ they will be wholly effective. $k$ is here taken as an indefinitely small portion of the surface. Let it be the element of a finite portion which is removed; then will the sum of the pressures on a portion of the vessel which has the same projection with the portion of it removed, be wholly effective on it in a direction perpendicular to the plane of projection.

Example. Let it be required to determine where a square aperture must be made in the side of an upright prismatic vessel of fluid, that it may just be overturned.

Let $pq$ (Fig. 14.) be a portion of the surface, of which the aperture $PQ$ is the projection. Then will the pressure sustained by the portion of surface removed from $PQ$, be wholly effective on $pq$, and will be the whole effective force on that surface.

Let $x_{\prime}$ be the distance of any point ($m$) in $PQ$ from $C$. Take $PQ = a$, $CP = x$, $AB = b$, $CA = c$. Therefore the pressure on an unit at $m = Dg(b - x_{\prime})$.

Momentum of the pressure on $PQ$ or on $pq$ about $A$,

$$= -Dga \int (b - x_{\prime})\, x_{\prime} dx_{\prime}$$

$$= -Dga \left\{ \left\{ \tfrac{1}{2} b (x + a)^2 - x^2 \right\} - \tfrac{1}{3} \left\{ (x + a)^3 - x^3 \right\} \right\};$$

taking the integral from $x_{\prime} = x$ to $x_{\prime} = x + a$,

$$= Dga \left\{ ax^2 + a^2 x + \frac{a^3}{3} - abx - \frac{bx^2}{2} \right\}.$$

Now, in order that when the aperture is first made, there may just be an equilibrium, this momentum must equal that produced about $A$ by the vertical pressure of the fluid.

Calling, therefore, the base of the vessel ($k$), we have

$$\tfrac{1}{2}Dgk.b.c = Dga\{(a-\tfrac{1}{2}b)x^2 + a(a-b)x + \tfrac{1}{3}a^3\};$$

$$\therefore\ x^2 + a\frac{a-b}{a-\frac{1}{2}b}x = \frac{kbc}{a(2a-b)} - \frac{a^3}{(3a-\frac{3}{2}b)};$$

$$\therefore\ x = -a\left(\frac{a-b}{2a-b}\right) \pm \sqrt{a^2\left(\frac{a-b}{2a-b}\right)^2 + \frac{3kbc-2a^4}{3a(2a-b)}}.$$

47. The center of pressure of the *whole* of any surface sustaining the pressure of a fluid, is determined by the intersection with that surface of a vertical line, passing through the center of gravity of the solid contained by the part of it immediately in contact with the fluid. For it has been shewn that the vertical pressure of the fluid is identical with the weight of such a solid; its resultant passes, therefore, through the center of gravity of that solid. Also the horizontal pressures destroy one another; the whole effective force is, therefore, the vertical pressure, and an equal and opposite pressure applied to the body anywhere in the direction of its resultant, will sustain it.

Hence, therefore, if a force equal to the weight of the fluid contained or the fluid displaced, be applied to the point where the vertical, through the center of gravity of the fluid contained or displaced, intersects the surface, it will sustain the body at rest. This point is the center of pressure.

48. Hence, it appears that if the surface sustaining the pressure of a fluid be symmetrical about a vertical axis, the intersection of that axis with the surface will be its center of pressure; and if that point be supported, the whole will be in equilibrio. Thus, a surface of revolution containing a fluid will be sustained by a horizontal plane, if placed on its vertex, &c. &c.

49. Let *ABCD* (Fig. 24.) represent a vertical section of a vessel containing a fluid in equilibrium.

Suppose a portion of the surface (projected in *AD*) to be curved *vertically*, so that each horizontal section may be

a straight line ($\lambda$) perpendicular to $ABCD$. It is required to find the magnitude and direction of the resultant of the forces on $AD$, tending to cause a motion of translation parallel to the plane $ABCD$. Take $Ox$, $Oy$, $Oz$, rectangular axes. Let $x, y, z$ be co-ordinates of any point. Now, calling $ds$ an element of the curve $AD$, $dxds$ is the corresponding element of the surface, and $Dgzdxds$ represents the normal pressure upon it. Now, the normal makes with the vertical an angle, whose $\cos = \frac{dy}{ds}$. Therefore, resolved in the directions of $z$ and $y$, the elementary pressures are respectively

$$Dgzdxdy \text{ and } Dgzdxdz.$$

Calling, therefore, $x_{,}$ and $y_{,}$ the distances of the resultant of the vertical forces from the planes $zy$ and $zx$, and $x_{,,}$ and $z_{,}$ the distances of the resultant of the horizontal forces from the planes $zy$ and $xy$, we have

$$x_{,} = \frac{\iint zxdxdy}{\iint zdxdy}$$

$$y_{,} = \frac{\iint zydxdy}{\iint zdxdy} = \frac{\int \lambda zydy}{\int \lambda zdy},$$

$$x_{,,} = \frac{\iint zxdxdz}{\iint xdxdz},$$

$$z_{,} = \frac{\iint z^2dxdz}{\iint zdxdz} = \frac{\int \lambda z^2dz}{\int \lambda zdz}.$$

The forces on $AD$ are, therefore, equivalent to two, acting parallel to the plane $ABCD$, one at the distance $x_{,}$ from that plane in a vertical direction, and the other acting at the distance $x_{,,}$ horizontally. Unless, therefore, $x_{,,} = x_{,}$, or the directions of these forces be in the same plane, no single force will sustain them and hold the system in equilibrium; or the pressures cannot be reduced to a single resultant. The resultant is represented by

$$Dg\{(\int \lambda zdy)^2 + (\int \lambda zdz)^2\}^{\frac{1}{2}}.$$

The co-ordinates of a point in it are $x_{,}$, $y_{,}$, and $z_{,}$, determined above; it is in a plane parallel to $ABCD$, and its direction makes with the vertical an angle, whose tangent is

$$\frac{\int \lambda z\, dy}{\int \lambda z\, dz}.$$

It is to be observed, that the resultant determined above does not represent a force equivalent to *all* the forces impressed upon the surface $AD$, but to those only which tend to produce a motion of translation parallel to $ABCD$.

50. Let us now consider the case in which the curvature is *horizontal*, and each vertical section a straight line.

Conceive $APB$ (Fig. 26.) to be any horizontal section of the vessel. The normal pressure on an element at $P$ is represented by $Dgz\,dz\,ds$, ($ds$ being an element of the curve $AB$). Resolving this in directions parallel to the planes $zx$ and $zy$, it becomes

$$Dgz\,dz \,.\, ds \,.\, \frac{dy}{ds} \text{ and } Dgz\,dz \,.\, ds \,.\, \frac{dx}{ds},$$

$$\text{or } \quad Dgz\,dz \,.\, dy \text{ and } Dgz\,dz \,.\, dx.$$

Taking, therefore, $y_{,}$ and $z_{,}$ for the distances of the resultant of the forces parallel to $zx$, from the planes $zx$ and $xy$ respectively; and $x_{,}$, $z_{,,}$ for the distances of the resultant of the forces parallel to $zy$, from the planes $zy$ and $xy$, we have

$$y_{,} = \frac{\iint zy\,dz\,dy}{\iint z\,dz\,dy},$$

$$z_{,} = \frac{\iint z^2\,dz\,dy}{\iint z\,dz\,dy},$$

$$x_{,} = \frac{\iint zx\,dz\,dx}{\iint z\,dx\,dz},$$

$$z_{,,} = \frac{\iint z^2\,dz\,dx}{\iint z\,dz\,dx}.$$

Now these resultants act in parallel (horizontal) planes and their directions are at right angles to one another. The system cannot, therefore, be sustained by any single force, unless $z_{,} = z_{,,}$.

By integration we get

$$x_{,} = \frac{\int z^2 x\,dx}{\int z^2 dx}, \qquad y_{,} = \frac{\int y z^2 dy}{\int z^2 dy},$$

$$z_{,} = \frac{\int y z^2 dz}{\int y z\,dz}, \qquad z_{,,} = \frac{\int x z^2 dz}{\int x z\,dz}.$$

Now, by the nature of the surface, $x$ and $y$ are independent of $z$;

$$\therefore\ x_{,} = \frac{z^2 \int x\,dx}{z^2 \int dx} = \tfrac{1}{2} x, \qquad y_{,} = \frac{z^2 \int y\,dy}{z^2 \int dy} = \tfrac{1}{2}(y + b),$$

$$z_{,} = \frac{y \int z^2 dz}{y \int z\,dz}, \qquad z_{,,} = \frac{x \int z^2 dz}{x \int z\,dz};$$

$$\therefore\ z_{,} = z_{,,} = \tfrac{2}{3}\left(\frac{z^3 - a^3}{z^2 - a^2}\right).$$

The surface being taken from the line of its intersection with the plane $zy$, $Oy$ being represented by $(b)$, and the pressure being supposed to commence when $z = a$.

Ex. To find at what depth an aperture $NB$, of given dimensions, must be made in a cylindrical vessel of fluid, that the effect to turn it over may be a maximum.

Let $h$ be the whole depth of the fluid. $NP = k$, $PB = a$, depth of $P = z$; therefore, the height of the center of pressure of $NB$ above the base of the cylinder is represented by

$$h - \tfrac{2}{3}\left(\frac{z^3 - a^3}{z^2 - a^2}\right) = h - \tfrac{2}{3}\left\{\frac{z^3 - (z-k)^3}{z^2 - (z-k)^2}\right\};$$

also the depth of the center of gravity of $NB = (z - \frac{1}{2}k)$; therefore, the pressure upon it is represented by $ka\,(z - \frac{1}{2}k)$

and the momentum tending to cause the body to revolve upon its base

$$= ka(z - \tfrac{1}{2}k)\left\{h - \tfrac{2}{3}\frac{3z^2k - 3k^2z + k^3}{2kz - k^2}\right\} = \text{max};$$

$$\therefore\ h(2z - k) - \tfrac{2}{3}(3z^2 - 3kz + k^2) = \text{max};$$

$$\therefore\ 2h - 4z + 2k = 0;$$

$$\therefore\ z = \tfrac{1}{2}(k + h).$$

We shall now take an *example* of the more general case of the center of pressure in curved surfaces.

*A sphere filled with water is divided vertically into two hemispheres: required the position and magnitude of the lateral forces which shall just prevent their separation.*

Let $(a)$ be the radius of a sphere, (Fig. 30.) $O$ its center, and $(x, y, z)$ the co-ordinates of a point $P$, in its surface. $O$ being the origin, and the plane $xy$ being horizontal. Now, the pressure on an element at $P$, resolved in a direction perpendicular to the plane $zy$, is equal to the weight of a column of the same depth with that point, and having for its base the projection of the element on $zy$; it is therefore equal to the weight of the column $(z + a)\,dzdy$. Similarly the pressure resolved perpendicular to $xy$, = the weight of the column $(z + a)\,dxdy$.

Calling, therefore, $z_{\prime}$ and $x_{\prime}$ the distances of the resultants of these parallel forces from the planes $xy$ and $zy$ respectively, we have

$$z_1 = \frac{\iint z\,.(z + a)\,dzdy}{\iint (z + a)\,dzdy},\ x_1 = \frac{\iint (z + a)\,xdxdy}{\iint (z + a)\,dxdy};$$

$$\iint (z^2 + az)\,dzdy = \int \{\tfrac{1}{3}z^3 + \tfrac{1}{2}az^2 + c\}\,dy,$$

$$= \int \frac{2\,(a^2 - y^2)^{\frac{3}{2}}}{3}\,dy, \text{ taken from } z = -\sqrt{a^2 - y^2}$$

$$\text{to } z = +\sqrt{a^2 - y^2},$$

$$= \tfrac{2}{3}\int (a^2-y^2)^{\frac{3}{2}}\, dy = \tfrac{2}{3}\left\{\frac{y\,(a^2-y^2)^{\frac{3}{2}}}{4}\right.$$

$$\left. + \frac{3a^2}{4}\cdot\left(\frac{y}{2}\sqrt{a^2-y^2} + \frac{a^2}{2}\sin^{-1}\frac{y}{a}\right)\right\} + c,$$

$$= \frac{\pi a^4}{4}, \text{ taken from } y = -a \text{ to } y = a;$$

$$\iint (z+a)\, dz\, dy = \int \{\tfrac{1}{2}z^2 + az + c\}\, dy,$$

$$= \int 2\,.\,a\,.\sqrt{a^2-y^2}\,.\,dy, \text{ taken from } z = -\sqrt{a^2-y^2} \text{ to } z = +\sqrt{a^2-y^2},$$

$$= 2a\left\{\tfrac{1}{2}y\sqrt{a^2-y^2} + \tfrac{1}{2}a^2\sin^{-1}\frac{y}{a}\right\} + c,$$

$$= \pi a^3 \text{ taken from } y = -a \text{ to } y = a;$$

$$\therefore\ z_1 = \frac{\pi a^4}{4\pi a^3} = \tfrac{1}{4}a.$$

$$\iint (z+a)\, x\, dx\, dy = \iint \{ax + x\sqrt{a^2-x^2-y^2}\}\, dx\, dy,$$

$$= \int \{\tfrac{1}{2}ax^2 - \tfrac{1}{3}(a^2-x^2-y^2)^{\frac{3}{2}} + c\}\, dy,$$

$$= \int \left\{\frac{a\,(a^2-y^2)}{2} + \tfrac{1}{3}(a^2-y^2)^{\frac{3}{2}}\right\} dy, \text{ taken from } x = 0 \text{ to } x = \sqrt{a^2-y^2},$$

$$= \tfrac{1}{2}a\left(a^2y - \frac{y^3}{3}\right) + \tfrac{1}{3}\left\{\frac{a\,(a^2-y^2)^{\frac{3}{2}}}{4} + \frac{3a^2y}{8}\times\sqrt{a^2-y^2} + \frac{3a^4}{8}\sin^{-1}\frac{y}{a}\right\},$$

$$= \frac{2a^4}{3} + \frac{\pi a^4}{8}, \text{ taken from } y = -a \text{ to } y = a.$$

$$\iint (z+a)\, dx\, dy = \iint \{a + \sqrt{a^2-x^2-y^2}\}\, dx\, dy,$$

$$=\int\left\{ax+\frac{x}{2}\sqrt{a^2-x^2-y^2}+\frac{a^2-y^2}{2}\times\right.$$

$$\left.\sin^{-1}\frac{x}{\sqrt{a^2-y^2}}+c\right\}dy,$$

$$=\int\left\{a\sqrt{a^2-y^2}+\frac{\pi}{2}\;\frac{a^2-y^2}{2}\right\}dy, \text{ taken from}$$

$$x=0 \text{ to } x=\sqrt{a^2-y^2},$$

$$=a\left\{\frac{y}{2}\sqrt{a^2-y^2}+\frac{a^2}{2}\sin^{-1}\frac{y}{a}\right\}+\frac{\pi}{4}\left(a^2y-\frac{y^3}{3}\right)+c,$$

$$=\frac{5\pi a^3}{6}, \text{ taken from } y=-a \text{ to } y=a;$$

$$\therefore\ x_1=\frac{\dfrac{2a^4}{3}+\dfrac{\pi a^4}{8}}{\dfrac{5\pi a^3}{6}}=a\left\{\frac{4}{5\pi}+\frac{3}{20}\right\}.$$

Also the resultant $=\sqrt{\left(\frac{5\pi a^3}{6}\right)^2+(\pi a^3)^2}=\frac{\pi a^3\sqrt{61}}{6}$.

Now, this resultant of the forces parallel to the axes of $z$ and $x$, is equal to the whole pressure effective in separating the hemispheres; and it manifestly acts in the plane $zx$, since the surface is symmetrical about that plane. Take, therefore, in the plane $zx$, a point determined by the co-ordinates $x_1$ and $z_1$. The intersection of the surface of the sphere, with a line drawn through this point and the center $O$, will be the center of pressure.

51. To determine the conditions of the equilibrium of an embankment or dyke.

Let $ABC$ (Fig. 28.) be a transverse section of any portion of the dyke, taken to be every where of uniform

dimensions. Now, let the dyke be supposed to be formed of solid materials liable only to revolve about the point $A$, or to slide along the horizontal line $BA$. Let $P$ be any point, $z$ its depth,

$$DN = NP = y, \quad AB = k, \quad BD = l.$$

By (Art. 49.) the vertical and horizontal pressures of the fluid on $P$ are respectively represented by

$$Dgzdxdy \text{ and } Dgzdxdz$$

and their momenta about $A$ are

$$Dgz\,(k-l+y)\,dydx, \text{ and } Dgz\,(h-z)\,dxdz.$$

Therefore, the whole momenta of the vertical and horizontal pressures of the fluid, are

$$Dg\iint z\,(k-l+y)\,dydx = Dg\lambda\int(k-l+y)\,zdy \text{ and}$$

$$Dg\iint z\,(h-z)\,dzdx = Dg\lambda\,(\tfrac{1}{2}h^3 - \tfrac{1}{3}h^3) = \tfrac{1}{6}Dg\lambda h^3,$$

taking the latter integral from 0 to $h$.

Now, let $M$ represent the area of the section $ABC$, $m$ the horizontal distance of its center of gravity from $A$, and $D'$ its density. Its momentum about $A$ will then be represented by

$$D'g \cdot \lambda \cdot M \cdot m$$

and, on the whole, that the equilibrium may obtain about $A$, we must have

$$Dg\lambda\int(k-l+y)\,zdy + D' \cdot g \cdot \lambda \cdot M \cdot m - \tfrac{1}{6}Dg\lambda h^3 = 0,$$

$$\text{or, } \int(k-l+y)\,zdy + \sigma Mm - \tfrac{1}{6}h^3 = 0,$$

representing by $\sigma$ the ratio of the specific gravities of the solid and fluid.

It remains for us now to consider the conditions necessary to prevent the dyke from sliding horizontally along the plane $BA$.

Its weight is $D'g\lambda M$; its friction on the plane $BA$, will therefore be represented by the product of this quantity, and a constant $(n)$ to be determined by experiment. Also, the horizontal pressure of the fluid is represented by

$$Dg\int\int z\,dx\,dz = \tfrac{1}{2}Dg\lambda h^2$$

The remaining condition of equilibrium will therefore be expressed by the equation,

$$D'g\lambda Mn - \tfrac{1}{2}Dg\lambda h^2 = 0$$

$$\text{or, } 2\sigma Mn - h^2 = 0 \ldots\ldots\ldots\ldots (2)$$

52. Let us apply these equations to the case in which the section $ABCD$ is a trapezium. (Fig. 15.) Let $EF = k'$ $CD = b'$ whole height of the dyke $= a$;

$$\therefore\ M = \tfrac{1}{2}a\,(k + k') \text{ also, by } \triangle^s\ \frac{y}{z} = \frac{l}{h},$$

$$\therefore \int (k - l + y)\,z\,dy = \frac{l}{h}\int\left(k - l + \frac{zl}{h}\right)z\,dz,$$

$$= l\,\frac{(k-l)}{2h}\,h^2 + \frac{l^2h^3}{3h^2}, \text{ taken from } 0 \text{ to } h$$

$$= \tfrac{1}{2}lkh - \tfrac{1}{6}\,l^2h;$$

$\therefore$ by substitution in equation (1),

$$\tfrac{1}{2}l\,.\,k\,.\,h - \tfrac{1}{6}\,l^2\,.\,h - \tfrac{1}{6}\,h^3 + \tfrac{1}{2}a\,(k + k')\,\sigma m = 0.$$

53. Let us now suppose any portion $CMP$ of the dyke $CAB$, (Fig. 18.) whose side $CA$ is vertical, to be liable to revolve about the point $M$. To determine the form of the curve $CPB$, in the case of equilibrium.

By equation (1) since in this case $l = k$,

$$\int yz\,dy - \tfrac{1}{6}\,z^3 + \sigma\,.\,M\,.\,m = 0,$$

where $M.m$ is the momentum of the area $CPM$ about $M$. Now the momentum of any ordinate $y$ about that point, equals $\frac{1}{2}y^2$; therefore on the whole, the momentum equals $\frac{1}{2}\int y^2 dz + A$, $A$ being the momentum of $CDE$ about the point $M$;

$$\therefore \int yzdy - \tfrac{1}{6}z^3 + \tfrac{1}{2}\sigma\int y^2 dz + \sigma A = 0;$$

therefore differentiating

$$yzdy - \tfrac{1}{2}z^2 dz + \tfrac{1}{2}\sigma y^2 dz = 0;$$

$$\therefore\ dy^2 + \sigma y^2 \frac{dz}{z} = z dz;$$

$$\therefore\ y^2 \epsilon^{\sigma\int\frac{dz}{z}} = \int \epsilon^{\sigma\int\frac{dz}{z}} z dz + C;$$

$$\therefore\ y^2 z^\sigma = \frac{z^{\sigma+2}}{\sigma + 2} + C;$$

$$\therefore\ y^2 = \frac{z^2}{\sigma + 2} + \frac{c}{z^\sigma}$$

when $z = 0$ $y = \infty^{ty}$. An embankment according to the proposed conditions cannot therefore be formed *from the very surface* of the fluid.

Let us now consider the case of an embankment sustaining a fluid of the class we have described as of imperfect fluidity.

Let the vertical surface $CB$, (Fig. 27.) sustain such a fluid mass, of earth, sand, or other imperfectly fluid substance. Now, it is observed, that if any portion $CP$ of this surface be removed, a mass, $CPM$, will detatch itself from the rest and roll down, leaving the surface $PM$, sensibly a plane, varying in its inclination to the vertical with the fluidity of the substance, being vertical in the case of a perfectly solid body, and horizontal in that of a perfect fluid. For

the same substance, whatever be the portion of surface removed, the angle $CPM$ is the same; let it be represented by $\phi$. Let $CP=x$ $CB=h$.

Now, the mass $CPM$ is sustained by the horizontal reaction of the surface $CP$, and by its friction on the surface $PM$, and its cohesion with it. Of these two last forces, the one is proportional to the vertical pressure on $PM$, and the other to the area of that surface. Call $Pg$ the horizontal pressure on $CP$. Resolved in the direction of $PM$, and perpendicular to that direction this becomes $Pg$ sin. $\phi$, and $Pg$ cos. $\phi$.

Also the weight of $CPM = \frac{1}{2}gx^2. \tan\phi \,.\, D$, and resolving this similiarity it becomes $\frac{1}{2}gDx^2 \sin.\phi$ and $\frac{1}{2}gDx^2 \tan.\phi \sin.\phi$, also $PM = x \sec.\phi$. Therefore, taking $\gamma$ for the coefficient of the cohesion and $(f)$ of the friction, we have

$$P\sin.\phi + Pf\cos.\phi + \tfrac{1}{2}gDx^2 f\tan\phi \,.\, \sin.\phi$$

$$+ \gamma x \sec.\phi - \tfrac{1}{2}gDx^2 \sin.\phi = 0;$$

$$\therefore\ P = \frac{\frac{1}{2}gDx^2 \sin.\phi - \frac{1}{2}gDx^2 f\tan.\phi \,.\, \sin.\phi - \gamma x \sec.\phi}{\sin.\phi + f\cos.\phi}.$$

Now, in a practical application of this formula, we may neglect the term $f$ cos. $\phi$, by which we shall favor the stability. Thus we shall obtain

$$P = \tfrac{1}{2}gDx^2(1 - f\tan.\phi) - \frac{\gamma x \sec.\phi}{\sin.\phi} \,\ldots\ldots (1);$$

differentiating equation (1), we obtain for the increment of horizontal pressure,

$$gDx(1 - f\tan\phi)\,dx - \frac{\gamma \sec\phi}{\sin\phi}\,dx;$$

The momentum therefore of the whole pressure, about $A$,

$$=\int (h-x)\left\{gDx\,(1-f\tan\phi)\,dx-\frac{\gamma\sec\phi}{\sin\phi}\,dx\right\}$$

which taken from $x = 0$ to $x = h$ gives

$$\tfrac{1}{6}gD\,(1-f\tan\phi)\,h^3-\tfrac{1}{2}\frac{\gamma\sec\phi}{\sin\phi}\,h^2,$$

calling therefore, $D'.M.m.g$, the momentum of $ABCD$;

$$D'.M.mg=\tfrac{1}{6}gD\,(1-f\tan\phi)\,h^3-\tfrac{1}{2}\frac{\gamma\sec\phi}{\sin\phi}\,h^2.$$

54. To determine the form of the curve $DQA$, any portion $CPQ$ of the dyke being supposed liable to revolve about $Q$. Let $PQ = y$. Now, the distance of a vertical through the center of gravity of $CPQ$ from $P$, equals

$$\tfrac{1}{2}\frac{\int y^2\,dx}{\int y\,dx};$$

and therefore from $Q$ it equals

$$y-\tfrac{1}{2}\frac{\int y^2\,dx}{\int y\,dx};$$

and therefore the momentum about

$$Q=y\int y\,dx-\tfrac{1}{2}\int y^2\,dx;$$

$$\therefore\ D'y\int y\,dx-\tfrac{1}{2}D'\int y^2\,dx=\tfrac{1}{6}D\,(1-f\tan\phi)\,x^3-\tfrac{1}{2}\frac{\gamma\sec\phi}{g\sin\phi}\,x^2$$

whence by differentiation and reduction we obtain

$$\left.\begin{aligned}&\left(\frac{d^2x}{dy^2}\right)\left\{\tfrac{1}{2}D'y^2-\tfrac{1}{2}D\,(1-f\tan\phi)\,x^2-\frac{\gamma\sec\phi}{g\sin\phi}\,x\right\}\\&-\left(\frac{dx}{dy}\right)^2\left\{D\,(1-f\tan\phi)\,x+\frac{\gamma\sec\phi}{g\sin\phi}\right\}+\left(\frac{dx}{dy}\right)2D'y\end{aligned}\right\}=0$$

by the solution of which equation the curve $DQA$ is determined.

*On the Surfaces of flexible Vessels containing Fluid.*

By a well known property of flexible curves, when the impressed force is perpendicular to the curve, (calling $T$ the tension and $R$ the radius of curvature) $T$ is constant; and the pressure at any point is represented by $\frac{T}{R}$.

55. Conceive a vessel to be formed of a series of horizontal flexible annuli. The pressure on every point in the circumference of each of these will be represented by the same quantity, viz. $zg$, where $z$ is the depth, and $g$ is the force of gravity, the density being unity;

$$\therefore\ R = \frac{T}{zg}.$$

The radius of curvature at every point in each annulus is therefore the same, or each annulus is a circle, and the whole, a surface of revolution.

If we take the case of a vertical plane curve sustaining the pressure of a fluid; as in the former case, we shall have

$$R = \frac{T}{zg};$$

$T$ being constant, and $z$ the variable ordinate to the curve. From the above equation, the nature of the curve may be determined.

56. To find the curve into which a flexible line will form itself, when sustaining at every point the pressure of a fluid mass, and acted upon by a force everywhere parallel to itself.

Let $P$ represent the pressure of the fluid, and $Q$ the force impressed on the curve at any point $P$, (Fig. 7.)

Let $\qquad AM = x, \qquad MP = y,$

the axis of $y$ being parallel to the direction of the force $Q$.

The pressure of the fluid is perpendicular to the curve; resolved therefore in the direction of $x$ and $y$ it becomes $P \cdot \frac{dy}{ds}$ and $P \cdot \frac{dx}{ds}$. The whole pressures on the element $ds$ resolved in these directions, are, therefore,

$$P \cdot \frac{dy}{ds} \cdot ds \text{ and } \left(P \frac{dx}{ds} + Q\right) ds,$$

$$\text{or} \quad P \cdot dy \quad \text{and } (Pdx + Qds).$$

Now, by a well known property of the funicular polygon, all the forces impressed on any branch $AP$ of the curve, if applied at $P$, would be in equilibrium with the tension ($T$) at that point.

Hence, therefore, $\int Pdy + T \frac{dx}{ds} = 0$..............(1)

$$\int (Pdx + Qds) - T \frac{dy}{ds} = 0 \text{..............(2).}$$

Differentiating the above equations, and multiplying the differential of the former by $dy$ and that of the latter by $dx$, and adding, we obtain

$$Pds^2 + Qdxds + T \frac{dyd^2x - dxd^2y}{ds} = 0.$$

Multiplying the differential of the former by $dx$, and of the latter by $dy$, and subtracting,

$$- Qdsdy + dT \cdot ds = 0;$$

or calling $R$ the radius of curvature,

$$\left(P + Q \frac{dx}{ds}\right) R + T = 0,$$

$$- Qdy + dT = 0.$$

Differentiating the former of these equations, eliminating $dT$, and reducing, we obtain the equation

$$PdR\,.\,ds + Q\,.\,dR\,.\,dx + RdPds + 2Qdyds + RdxdQ = 0.$$

Suppose the force $P$ to be constant, and $Q$ to vanish, as when the curve is horizontal and the fluid acted upon by the force of gravity. Then, since $Q = 0$ and $dP = 0$,

$$PdR\,.\,ds = 0;$$

$$\therefore\ dR = 0:$$

therefore $R$ is constant, or the curve is a circle.

Next, let the force $Q$ be supposed to vanish,

$$PdRds + RdPds = 0;$$

$$\therefore\ PR = C.$$

If the curve be taken in a vertical plane, and the force $P$ be that of gravity, we have $P = \sigma g y$;

$$\therefore\ gRy = C,$$

$$\text{or} \qquad y \propto \frac{1}{R}.$$

If $Q$ be considered constant, as, for instance, the weight of an element of the containing surface $= mg$, the general equation becomes

$$\sigma ydRds + mdRdx + \sigma Rdyds + 2mdyds = 0.$$

Now, $ds$ being constant,

$$R = \frac{dyds}{d^2x}; \qquad \therefore\ dyds = Rd^2x;$$

$$\therefore\ \sigma yd\,Rds + \sigma\,Rdyds + mdRdx + mRd^2x + mdyds = 0.$$

Integrating on the supposition that $(ds)$ is constant,

$$\sigma\,R\,yds + mRdx + myds = Ads;$$

eliminating $R$,

$$\sigma y dy ds + m dx dy + m y d^2 x = A d^2 x;$$

$$\therefore \tfrac{1}{2}\sigma y^2 ds + m y dx = A dx + B ds:$$

from whence we readily obtain

$$dx = \frac{(\frac{1}{2}\sigma y^2 - B)\, dy}{\sqrt{(A - my)^2 - (\frac{1}{2}\sigma y^2 - B)^2}}.^*$$

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* This particular case may be solved immediately from the equations (1) and (2). Substituting the values of $P$ and $Q$, we get

$$g\sigma \int y dy + T\frac{dx}{ds} = 0,$$

and $$g\int (\sigma y\, dx + m ds) - T\frac{dy}{ds} = 0.$$

Performing the integration indicated in the former equation, we have

$$g(\tfrac{1}{2}\sigma y^2 - B) + T\frac{dx}{ds} = 0:$$

the constant $B$ being determined by the value of the tension and the direction of the curvature, when $y = 0$.

Eliminating $T$, $(\frac{1}{2}\sigma y^2 - B)\, dy + dx \int (\sigma y dx + m ds) = 0$

$$\tfrac{1}{2}\sigma y^2 y' - By' + \int \left\{ \sigma y dx + m\frac{(1+y'^2)^{\frac{1}{2}}\, dy}{y'} \right\} = 0;$$

$\therefore$ differentiating

$$(\tfrac{1}{2}\sigma y^2 - B)\, dy' + \sigma\frac{(1+y'^2)\, y\, dy}{y'} + \frac{m(1+y'^2)^{\frac{1}{2}}}{y'}dy = 0;$$

$$\therefore \frac{y' dy'}{(1+y'^2)^{\frac{1}{2}}} + \sigma\frac{(1+y'^2)^{\frac{1}{2}}\, y\, dy}{(\frac{1}{2}\sigma y^2 - B)} + \frac{m dy}{(\frac{1}{2}\sigma y^2 - B)} = 0;$$

a linear equation,

$$\therefore (1+y'^2)^{\frac{1}{2}} = -m.\epsilon^{\int \frac{-\sigma y dy}{(\frac{1}{2}\sigma y^2 - B)}} \left\{ \int \epsilon^{\int \frac{\sigma y dy}{(\frac{1}{2}\sigma y^2 - B)}} \frac{dy}{(\frac{1}{2}\sigma y^2 - B)} + C \right\}$$

$$= \frac{-m(y+A)}{(\frac{1}{2}\sigma y^2 - B)}$$

$$dx = \frac{(\frac{1}{2}\sigma y^2 - B)\, dy}{\{m^2(y+A)^2 - (\frac{1}{2}\sigma y^2 - B)^2\}^{\frac{1}{2}}}$$

$$T = mg(y+A).$$

A third constant will be introduced by this last integration. To determine these constants, we may observe, that when $y=0$, $x=0$; and that the length of the curve and the distance of the points of suspension are given.

If we make $\sigma=0$,

$$dx = \frac{-Bdy}{\sqrt{(A-my)^2-B^2}},$$

which is the equation to the common catenary.

If $m=0$,

$$dx = \frac{\frac{1}{2}\sigma y^2 dy}{\sqrt{A^2-(\frac{1}{2}\sigma y^2-B)^2}};$$

which is the common equation to the velinary curve, the weight of the containing surface being neglected.

## CHAP. V.

### ON THE EQUILIBRIUM OF FLOATING BODIES.

57. A HEAVY body immersed in a fluid being pressed downwards by forces, the sum of which is represented by its weight, and upwards by forces whose sum is equal to the weight of the fluid displaced, cannot be sustained; first, unless the sums of these forces when thus severally taken together be equal to one another; and secondly, unless the directions of their resultants coincide*.

* Since it is a necessary condition to the equilibrium of a system acted upon by any number of forces, that the sum of those forces when estimated in any given direction, should be zero; and that being compounded into any two resultants, the directions of these should be in the same straight line.

From the first condition, it follows that the weight of a floating body equals that of the fluid it displaces.

From the second, that (since the vertical pressures on the different points of the surface immersed, are precisely analogous to the weights of a system of columns, forming together the part of the solid which is immersed, and the resultants of the two therefore coincide); the center of gravity of the body and of the part of it immersed, are in the same vertical line.

These conditions are also *sufficient* to the equilibrium, for they manifestly establish it with regard to the vertical forces on the system, and it has been shewn that the horizontal pressures respectively destroy one another in every position of the body.

58. Let $M$ be the volume of a solid, $D$ its density, $M'$ the volume of the part of it immersed when it floats in equilibrium, in a fluid whose density is $D'$.

Then, since $MDg$ is the weight or downward pressure of the body, and $M'D'g$, that of the fluid it displaces, we have by the first condition of equilibrium

$$M \cdot D = M' \cdot D' \dots\dots\dots\dots\dots (\eta)$$

If the body be wholly plunged in the fluid, $M' = M$, and, therefore, $D' = D$.

If it be only partially immersed, $M' < M$; and, therefore, $D' > D$.

It is, therefore, in all cases necessary to the equilibrium, that the density of the body be *not less* than that of the fluid in which it is immersed.

59. Generally the forces by which the body is urged upwards and downwards are respectively represented by $MD$ and $M'D'$; it will therefore ascend, or remain at rest, or descend, according as

$$M'D' > = < MD;$$

that is, if the body be wholly plunged, according as

$$D' > = < D.$$

In the first case it will ascend until a portion of it has emerged, and the equality $M'D' = MD$ being at length established, the moving force ceases and it eventually floats at rest. In the last case it will descend continually.

If the body be placed on the surface of the fluid, and $D' = D$, it will descend until the whole of it is immersed, when $M'D'$ being equal to $MD$, the moving force will cease; and being projected with the velocity acquired in its descent, it will proceed in the direction of projection until its motion has been wholly destroyed by the resistance of the fluid, and it rests suspended.

If $D' > D$, the equality $M'D' = MD$ will be established, and the moving force destroyed, before the body is entirely immersed. By the velocity acquired in its descent, it will however be projected beyond the position of equilibrium, and $M'D'$ becoming $< MD$, a velocity will eventually be generated in an opposite direction, and the body will oscillate on the surface until all motion is destroyed by the continual resistance of the fluid.

60. If the fluid be of uniform density, and $D$ be taken to represent the *mean* density of the body, so that $MDg$ may represent its weight as before, we shall have

$$MD = M'D'.$$

If the floating body be hollow, the mean density is that quantity which being multiplied by its bulk, taken externally, will equal its weight.

If, therefore, it be admitted that a given portion of material can be so disposed as to be of any given external bulk, then, (so far as this condition of equilibrium is concerned,) it appears that any such portion of matter can be made to float. It is a further condition of the equilibrium, that the *solid* con-

tained by the surface actually immersed, should have its center of gravity in the same vertical line with that of the vessel.

61. If the body be symmetrical about a certain axis, that is, if it be such, that being cut transversely by any plane perpendicular to that axis, the center of gravity of the section may lie in the axis; it is evident that the center of gravity of the whole solid, and of either of the parts cut off, will also lie in the axis. If, therefore, the solid be immersed with its axis vertical, the center of gravity of the whole and of the part immersed, will lie in the same vertical, viz. the axis; and the second condition of equilibrium will be satisfied to whatever depth it be sunk.

Ex. Let the body immersed be a sphere of uniform density, it is evident from what has been said above, that it will float in any position; now by the first condition of equilibrium, we have, if $AM$ (Fig. 16.) $= x$, $PM = y$ ($P$ being the lowest point of immersion,) rad. $= a$

$$\begin{aligned} \tfrac{4}{3} \cdot D\pi a^3 &= D'\pi \textstyle\int y^2 dx \\ &= D'\pi \textstyle\int (2ax - x^2)\, dx \\ &= D'\pi \left(ax^2 - \tfrac{1}{3}x^3\right); \\ \therefore\ 4\frac{D}{D'}a^3 &= 3ax^2 - x^3, \end{aligned}$$

or, if $\sigma =$ the ratio of the specific gravities of the solid and fluid, which is the same with that of their densities, we have

$$x^3 - 3ax^2 + 4 \cdot \sigma \cdot a^3 = 0.$$

By the solution of which cubic equation $x$ is known.

It is manifest, that, since, as the body descends, the quantity of fluid it displaces increases, whilst its weight remains the same; there can but be one position thus found, in which the weight of the one can equal that of the other.

If the sphere be loaded with a weight $(u)$, we shall have

$$\tfrac{4}{3} D\pi a^3 + u = D'\pi\,(a x^2 - \tfrac{1}{3} x^3)$$

to simplify the expression, suppose the weight to equal that of a sphere, of the same density with the solid whose radius is $a$;

$$\therefore\ \tfrac{4}{3} D\pi a^3 + \tfrac{4}{3} D\pi a^3 = D'\pi(a x^2 - \tfrac{1}{3} x^3);$$

$$\therefore\ \tfrac{4}{3}\sigma\,(a^3 + a^3) = a x^2 - \tfrac{1}{3} x^3.$$

Now, let it be required to determine what must be the value of $(a)$, that $x$ may be a minimum. Or in other words, to find that sphere of a given density which will support a given weight, and sink to the least possible depth. Differentiating with respect to $(a)$,

$$4a^2\sigma = x^2 + 2ax\frac{dx}{da} - x^2\frac{dx}{da},$$

and making $\dfrac{dx}{da} = 0$, since $x$ is a minimum,

$$x = \pm 2a\sqrt{\sigma}.$$

Differentiating again,

$$8a\sigma = 2x\frac{dx}{da} + 2x\frac{dx}{da} + 2a\left(\frac{dx}{da}\right)^2$$

$$+ 2ax\frac{d^2x}{da^2} - 2x\left(\frac{dx}{da}\right)^2 - x^2\frac{d^2x}{da^2},$$

and substituting for $x$,

$$8a\sigma = \frac{d^2x}{da^2}(\pm 4a^2\sqrt{\sigma} - 4a^2\sigma);$$

$$\therefore\ \frac{d^2x}{da^2} = \frac{2\sqrt{\sigma}}{a(\pm 1 - \sqrt{\sigma})}$$

Now $\sigma$, and therefore $\sqrt{\sigma}$ is essentially less than unity, therefore $\frac{d^2 x}{d a^2}$ is positive or negative, according as the positive or negative sign is taken. The positive sign, therefore, corresponds to a minimum, and we have

$$\tfrac{4}{3}\sigma(a^3 + a^3) = 4a^3\sigma - \tfrac{8}{3}a^3\sigma^{\frac{3}{2}};$$

$$\therefore\ 8a^3(1 - \sigma^{\frac{1}{2}}) = 4a^3;$$

$$\therefore\ a = \frac{a}{\sqrt[3]{2(1 - \sigma^{\frac{1}{2}})}}.$$

Ex. 2. Having given the whole quantity of the materials and lading of a rectangular vessel, the part of which, in contact with the fluid in which it floats, is to be cased with a given surface of copper, it is required to determine the dimensions of this last part, that the depth to which the vessel sinks may be a maximum.

Let $x$ and $y$ be the edges of the base of the vessel, and $z$ the depth of immersion. Then is the surface immersed represented by

$$xy + 2zx + 2zy.$$

If, therefore, ($c$) be the given surface of copper,

$$xy + 2zx + 2zy = c.$$

Also, if ($m$) represent the whole quantity of material and lading, and $D$ its mean density,

$$xyzD' = mD,$$

$$xyz = m\sigma;$$

$$\therefore\ \frac{m\sigma(x + 2z)}{zx} + 2zx = c,$$

$$\text{or,}\qquad m\sigma\left(\frac{1}{z} + \frac{2}{x}\right) + 2zx = c.$$

Now, since $z$ is a maximum,

$$\frac{dz}{dx} = 0;$$

$$\therefore\ -\frac{2m\sigma}{x^2} + 2z = 0;$$

$$\therefore\ z = \frac{m\sigma}{x^2};$$

whence by substitution and reduction,

$$x^3 - cx + 4m\sigma = 0;$$

the roots of which equation determine the maximum and minimum values of the function.

All the roots are possible if $c > 3(2m\sigma)^{\frac{2}{3}}$, two are impossible if this be not the case.

Ex. 3. In the above example the quantity of material and lading being given, it is required to determine the dimensions of the vessel, that the copper sheathing to be used, or the surface exposed to the action of the fluid, may be a minimum.

Here $$xyz = m\sigma,$$

and $$xy + 2xz + 2gz = \text{minimum};$$

therefore, $$m\sigma\left(\frac{1}{z} + \frac{2}{x}\right) + 2zx = \text{minimum};$$

therefore, differentiating with respect to $z$ and $x$,

$$-\frac{m\sigma}{z^2} + 2x = 0,$$

$$-\frac{2m\sigma}{x^2} + 2z = 0;$$

therefore by substitution,

$$\frac{x^4}{m\sigma} + 2x = 0,$$

$$-(2m\sigma)^{\frac{1}{3}} + 2z = 0,$$

$$\tfrac{1}{2}(2m\sigma)^{\frac{2}{3}} y - m\sigma = 0;$$

$$\therefore\ x = (2m\sigma)^{\frac{1}{3}},$$

$$z = \tfrac{1}{2}(2m\sigma)^{\frac{1}{3}},$$

$$y = (2m\sigma)^{\frac{1}{3}}.$$

The base of the vessel is therefore a square, and the depth to which it sinks is equal to one half the side of its base.

Ex. 4. To find to what depth an ellipsoid will sink, with its major axis in a vertical position.

Let $a$, $b$, $c$ be the semi axes of the ellipsoid, $x$, $y$, $z$ the co-ordinates of any point in it, and $x$ the distance of the surface of the fluid from its center;

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1.$$

Now, a horizontal section of the figure through the point $(x, y, z)$ is an ellipse, in which $y$, $z$ are the co-ordinates of that point from its center. Also,

$$\frac{y^2}{\frac{b^2}{a^2}(a^2 - x^2)} + \frac{z^2}{\frac{c^2}{a^2}(a^2 - x^2)} = 1;$$

therefore $\frac{b}{a}(a^2 - x^2)^{\frac{1}{2}}$, and $\frac{c}{a}(a^2 - x^2)^{\frac{1}{2}}$

are the semi axes of this ellipse;

$$\text{and } \frac{\pi bc}{a^2}(a^2 - x^2)$$

is its area:

$$\text{therefore, the part immersed} = \frac{\pi bc}{a^2}\int(a^2 - x^2)\,dx$$

$$= \frac{\pi bc}{a^2}\left\{a^2(a - x_,) - \tfrac{1}{3}(a^3 - x_,^3)\right\},$$

taking the integral from $x_,$ to $a$.

Now the whole content of the ellipsoid is

$$\tfrac{4}{3}\pi abc;$$

$$\therefore \frac{\pi bc}{a^2}\left\{a^2(a - x_,) - \tfrac{1}{3}(a^3 - x_,^3)\right\} = \tfrac{4}{3}\pi abc\sigma;$$

$$\therefore x_,^3 - 3a^2x_, - 2a^3(2\sigma - 1) = 0;$$

which equation not involving $b$ and $c$, it is clear that $x_,$ is independent of those quantities, and therefore the same for a given value of $(a)$, whatever relation exist between them, or, it is the same for the ellipsoid, spheroid, and sphere.

It is clear that the above belongs to what is termed the irreducible case of the cubic equation, since

$$\left\{2a^3(2\sigma - 1)\right\}^2 - \frac{4(3a^2)^3}{27} = 16a^6\sigma(\sigma - 1,)$$

and that $\sigma$ being essentially less than unity, this expression is negative. Hence, therefore, it appears that all its roots are real.

To obtain them, let

$$\cos\phi = \frac{2a^3(2\sigma_, - 1)}{2a^2\sqrt{a^2}}* = 2\sigma - 1,$$

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* See Francœur Cours de Mathematics, Art. 550.

$$\text{then,} \quad x_{,} = 2a \cos \tfrac{1}{3}\phi,$$

we shall obtain the *three* values of $x_{,}$ by substituting in this last equation, the quantities $\phi$, $\phi + 2\pi$, $\phi + 4\pi$, obtained from the former. It is clear that but *one* of these roots can answer the conditions of the question. The two others give values of $x_{,}$ greater than $a$.

Thus, if we take $\sigma = \frac{1}{2}$, the values of $\phi$ are

$$\frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2},$$

and the corresponding values of $x_{,}$ are $a\sqrt{3}$, $-a\sqrt{3}$, 0, of which the two former are greater than $(a)$. The last is manifestly that which solves the problem.

Where a given weight is to be sustained, it is clear that a given quantity of fluid must be displaced. Of all floating bodies therefore, that, the part of which immersed is of a spherical form, will sustain a given weight exposing the least surface to the action of the fluid.

62. If the fluid be of unlimited extent, it is evident that the portion of it displaced by a vessel of finite magnitude, can have no sensible effect in increasing the altitude of its surface. Whereas, if the volume of the fluid displaced be finite, as compared with its whole mass, (or rather, as compared with that portion of it which lies above a horizontal plane, passing through the lowest point of the body immersed), then will the immersion have a finite effect in elevating the surface, and consequently in altering the position of the plane of flotation.

Ex. To find the depth to which a paraboloid of revolution will sink in a cylindrical vessel of fluid.

Let $D$ and $D'$ be the densities of the solid and fluid. $K$ the area of the base of the cylinder, $K'$ that of the base of the paraboloid. $LN$, (Fig. 17.) $= a$. $DC$, the altitude

at which the fluid stood before the immersion, $= b$. The area of the plane of flotation $PQ = k$, $MN = a$, $DP' = x$.

Now, the volume of the paraboloid $AN$

$$= \tfrac{1}{2}K'a, \text{ that of } PN = \tfrac{1}{2}ka;$$

therefore, by the first condition of equilibrium,

$$\tfrac{1}{2}DK'a = \tfrac{1}{2}D'ka;$$

and since the whole quantity of fluid is the same as before the immersion,

$$Kx - \tfrac{1}{2}ka = Kb;$$

$$\text{therefore, } KD'x - \tfrac{1}{2}kD'a = KD'b,$$

$$\text{and therefore, } KD'x - \tfrac{1}{2}K'Da = KD'b;$$

$$\text{therefore, } x = b + \tfrac{1}{2}a \cdot \frac{K' \cdot D}{K \cdot D'}.$$

All that has been stated with regard to the conditions of the equilibrium of floating bodies, applies equally whether the density of the body or fluid be uniform or not. In the case of variable density, the weight of the fluid displaced is represented by $\int D'dM'$; and that of the solid, by $\int DdM$. The second condition becomes, therefore,

$$\sigma \cdot \int DdM = \int D'dM';$$

the one integral being taken throughout the solid, and the other, with regard only to that portion of it which is immersed.

Ex. How deep will a sphere of given uniform density sink, when immersed in a fluid whose density varies as its depth $B$, (Fig. 16.) the surface of the fluid; $C$ the center of the sphere.

$$BM = x, \; MP = y, \text{ radius} = a.$$

Now, the upward pressure on the body, equals the weight of the fluid; which, if it were removed, would occupy the space at present filled by it.

Take the sphere itself to represent this portion of the fluid, then $\pi y^2$ is a horizontal section; and $\pi y^2 dz$ an element of it. Also, the density of this section is everywhere the same. Let it be represented by $ax$, where $a$ = density at distance unity;

$$\therefore \text{ weight of element } = \pi g a y^2 x dx;$$

whole weight of fluid displaced $= g\pi a \int y^2 x dx$.

Now if $BC = k$,

$$(k - x)^2 + y^2 = a^2;$$

$$\therefore \text{ weight } = g a \pi \int \{a^2 - (k - x)^2\} \, x dx.$$

Integrating this equation between the limits $(k + a)$ and $(k - a)$, we get for the whole weight of the fluid displaced,

$$\tfrac{4}{3} g \pi a^3 a k.$$

Now if $\beta$ be the density of the solid, its weight is

$$\tfrac{4}{3} g \pi a^3 \beta;$$

when therefore there is an equilibrium,

$$\tfrac{4}{3} \pi a^3 a k = \tfrac{4}{3} \pi a^3 \beta;$$

$$\therefore \; k = \frac{\beta}{a}.$$

That is, $k$ equals the ratio of the specific gravity of the solid to that of the fluid, at distance unity. Since $\beta = ka$ = density of the fluid about the center $c$ of the sphere, it appears that the equilibrium will take place when the density of the sphere is the same with that of the fluid about its center.

The above is a particular case of the following *general* property. A body, *of any form whatever*, will be in equilibrium in a fluid whose density varies as the distance from its surface, when it is immersed to such a depth, that its density is equal to that of the fluid about its center of gravity. Taking $dM$ to represent an element of the volume of the fluid displaced, at the depth $z$, its density will be represented by $az$, and therefore the quantity of matter displaced by $azdm$; therefore, the whole quantity of matter displaced, equals $a\int zdM$; or, calling $z_,$ the depth of the center of gravity, we have for the whole quantity of fluid matter displaced, $az_,M$;

$$\therefore \; az_,M = \beta \cdot M;$$

$$\therefore \; az_, = \beta.$$

Now, $az_,$ represents the density about the center of gravity of the body. Therefore, the density of the fluid about the center of gravity of the body, is equal to that of the body itself.

It remains for us now to consider some of the cases involving both conditions of equilibrium.

63. If the body be prismatic or cylindrical, (or in other words, if it be generated by the motion of a plane surface perpendicular to itself), and immersed so that its axis may be horizontal, or its two extremities in a vertical position; it is evident that into whatever position it be turned, its center of gravity and that of the part of it immersed lie in the same vertical section, and in fact coincide with the center of gravity of *that section* and the part of *it* immersed. And further, the ratio of the mass of the whole to that of the part immersed is the same in both cases, the conditions of equilibrium are therefore the same in every respect. And the determination of the positions of the equilibrium of a body taken as above, reduces itself to the determination of the positions of equilibrium of one of its generating sections.

64. Ex. Let us take the case of a triangular prism, and consider one of its angles as immersed; by what has been said before, it appears that the conditions of equilibrium are the same as in the triangle.

Let $ABC$, (Fig. 21.) be a triangular section of the prism. Now, it may occur that *one*, or *two* of the vertices are immersed; we shall deduce the conditions in the latter case from those of the former.

Let $a$, $b$, $c$ be the sides of the triangle opposite to the angles $A$, $B$, $C$; and $x, y$ the sides $AP$ and $AQ$ of the part of it immersed;

$$\therefore\ \triangle ABC = \tfrac{1}{2}bc \,.\, \sin A,$$

$$\triangle AQP = \tfrac{1}{2}xy \,.\, \sin A;$$

$$\therefore\ \tfrac{1}{2}bc \,.\, \sin A \,.\, D = \tfrac{1}{2}xy \,.\, \sin A \,.\, D';$$

or, if $\sigma =$ the ratio of the specific gravities, and therefore of the densities.

$$bc\sigma = xy \,\ldots\ldots\ldots\ldots\ldots\, (1).$$

Bisect the bases $BC$ and $PQ$, in $M$ and $m$; and take $MM'$ equal to one third of $MA$, and $mm'$ to one third of $mA$, then are $M'$ and $m'$ the centers of gravity of $ABC$ and $APQ$ respectively. And when there is an equilibrium, the line $M'm'$ is vertical or perpendicular to $PQ$.

Now, since $AM$ and $Am$ are divided in $M'$ and $m'$, in the same ratio it follows that $Mm$ is parallel to $M'm'$, and therefore perpendicular to $PQ$; hence $MP$ and $MQ$ are equal. And reciprocally if these lines be equal; $Mm$, and its parallel $M'm'$ are perpendicular to $PQ$. It is therefore necessary and sufficient to the equilibrium, that $MP$ should equal $MQ$. Now, if $MAP = \beta$, $MAQ = \gamma$, and $AM = h$,

$$MP^2 = h^2 - 2hx\cos\beta + x^2,$$

$$MQ^2 = h^2 - 2hy\cos\gamma + y^2;$$

$$\therefore\ x^2 - 2hx\cos\beta = y^2 - 2hy\cos\gamma \,\ldots\ldots\ldots\, (2).$$

The equations (1) and (2) determine the quantities $x$ and $y$, and involve the solution of the problem. Eliminating ($y$) and reducing, we obtain

$$x^4 - 2h \cdot \cos\beta\, x^3 + 2bch\sigma \cdot \cos\gamma \cdot x - \sigma^2 b^2 c^2 = 0.$$

or, since $h = b\cos\beta = c\cos\gamma$;

$$\therefore\ x^4 - 2b\cos^2\beta \cdot x^3 + 2bc^2\sigma\cos^2\gamma \cdot x - \sigma^2 b^2 c^2 = 0 \ldots\ldots (3).$$

If the two angles $B$ and $C$ be immersed, since the centers of gravity, of the whole triangle $ABC$, and its parts $BCPQ$ and $APQ$ are in the same straight line, and that the line joining the two first points is perpendicular to $PQ$, it follows that the line $M'm'$ joining the first and last, is also perpendicular to $PQ$. The equation (2) therefore remains; $x$ and $y$ being taken to represent the distances $AP$ and $AQ$, measured from the angle which is not immersed.

Also, $\overline{ABC} \cdot \sigma = \overline{BQPC}$,

$$= \overline{ABC} - \overline{AQP};$$

$$\therefore\ \overline{AQP} = \overline{ABC} \cdot (1 - \sigma);$$

$$\text{or, } xy = b \cdot c \cdot (1 - \sigma),$$

which differs from equation (1) in this, that $(1 - \sigma)$ is substitute for $(\sigma)$. With this change therefore, the elimination will be the same as that of the equations (1) and (2), and we have

$$x^4 - 2b\cos^2\beta x^3 + 2bc^2(1-\sigma)\cos^2\gamma \cdot x - (1-\sigma)^2 b^2 c^2 = 0 \ldots (4).$$

Now, every equation of even dimensions whose last term is negative, has at least two possible roots, of which one is positive and the other negative, the equation has therefore at least one positive root; the two remaining roots may be real or imaginary. If the four roots of the equation are real, it follows according to Descarte's rule of signs, that three of them are positive and the fourth negative; for whether we suppose the evanescent term of the equation $(O \cdot x^2)$ to have the sign + or −, we shall find three changes, and one

continuation of sign. Hence it follows, since the values of $x$ and $y$ are essentially positive, that at the utmost the equation cannot indicate more than three roots corresponding to different positions of equilibrium.

The roots satisfying the conditions are further limited to such as being positive and less than $b$, give by substitution values of $y$, which are also positive and less than $c$. Since it is manifestly necessary that $x$ and $y$ should be less than $b$ and $c$ respectively.

What has been said with regard to the number of possible roots, applies to both the equations (3) and (4). It appears therefore, that for every angle and for every two angles of the triangle, there are three; and therefore on the whole, eighteen possible positions of equilibrium.

Let us take the case of an equilateral triangle. Here

$$\beta = \gamma = 30^0, \text{ and } a = b = c;$$

therefore, by equation (2), we have

$$(x^2 - y^2) - 2h\cos 30\,(x - y) = 0,$$

$$(x - y)\left\{x + y - 2a\cos^2 30\right\} = 0,$$

$$(x - y)\left\{x + y - \frac{3a}{2}\right\} = 0;$$

$$\text{also by (1),} \qquad xy = \sigma a^2.$$

These equations are satisfied by taking

$$x = y = a\sqrt{\sigma};$$

and since $\sigma$ and therefore $\sqrt{\sigma}$ is $< 1$, this value of $x$ and $y$ is possible according to the conditions of the question, and indicates an actual position of equilibrium. The equations are farther satisfied by taking

$$x + y = \frac{3a}{2},$$

$$xy = a^2\sigma,$$

whence we obtain values of $x$ and $y$ alternately represented by the formula

$$\frac{a}{4}\{3 \pm \sqrt{9 - 16\sigma}\},$$

in the case of two angles being immersed, this becomes

$$\frac{a}{4}\{3 \pm \sqrt{9 - 16(1 - \sigma)}\};$$

$$\text{or,} \quad \frac{a}{4}\{3 \pm \sqrt{16\sigma - 7}\}.$$

Now, that the values of the first formula may be real and less than $(a)$, it is necessary first that $16\sigma$ should be

$$< = 9 \text{ or } \sigma < = \frac{9}{16},$$

and secondly that

$$\sqrt{9 - 16\sigma} \text{ should be } < = 1 \text{ or, } \sigma = > \tfrac{1}{2},$$

since if this last condition be not satisfied,

$$3 + \sqrt{9 - 16\sigma}$$

will exceed 4, and one of the quantities $x$ and $y$ will be greater than $(a)$. The limits therefore of the values of $\sigma$, so that the equilibrium may obtain in an oblique position of the triangle, and when one angle is immersed, are

$$\frac{1}{2} \text{ and } \frac{9}{16}$$

and similarly the limits when two angles are immersed, are

$$\frac{7}{16} \text{ and } \frac{1}{2}.$$

In order that the whole eighteen possible positions of equilibrium may therefore obtain, we must have $\sigma = \frac{1}{2}$, and $\sigma$ being without the limits

$$\frac{7}{16} \text{ and } \frac{9}{16},$$

the triangle will not rest in any other than its vertical positions.

65. An irregular area floats upon a fluid, the part immersed being triangular. It is required to determine the boundary of that portion of it which, in its revolution about its center of gravity, will always be above the surface of the fluid: or, in other words, to find the curve to which the line of flotation will always be a tangent.

In any position of the area $KACB$, (Fig. 19.) let $PQ$ be the line of flotation. Draw $MN$ parallel to $CB$. Let the whole area $=A$, $CM=x$, $MN=y$, $AC=a$, $BC=b$, $CP=\alpha$, $CQ=\beta$; and let $\sigma$ represent the ratio of the specific gravity of the solid to that of the fluid.

$$\text{By } \triangle^{s}, \quad \frac{\alpha - x}{y} = \frac{\alpha}{\beta} \ldots\ldots\ldots\ldots\ldots (1).$$

Also by the first condition of equilibrium, which is satisfied in every position of the body, since no vertical motion is supposed to take place,

$$\frac{1}{2}\alpha\beta \sin C = A\sigma;$$

$$\therefore \alpha\beta = \frac{2A\sigma}{\sin C} = 4c^2, \text{ suppose,}$$

$$\therefore \frac{\alpha}{\beta} = \frac{\alpha^2}{4c^2}; \qquad \therefore \frac{\alpha - x}{y} = \frac{\alpha^2}{4c^2}.$$

Differentiating (by the method of parameters) with regard to the arbitrary constant $\alpha$,

$$\frac{1}{y} = \frac{\alpha}{2c^2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

$$\therefore \alpha = \frac{2c^2}{y};$$

and by substitution in equation (1),

$$\frac{2c^2}{y^2} - \frac{x}{y} = \frac{c^2}{y^2};$$

$$\therefore xy = c^2,$$

the known equation to a rectangular hyperbola. By elimination in equation (1), we obtain $x = \frac{1}{2}a$; therefore by similar triangles,

$$QN = \frac{1}{2} PQ.$$

66. To determine the positions of the equilibrium of an area taken as above.

*AC* and *BC* being taken as before for the axes of $x$ and $y$, let $\phi$ be the inclination of the normal at $N$, or of the vertical, to $CA$. Also, let $x_{\prime\prime}$ and $y_{\prime\prime}$ be the rectangular co-ordinates of $N$;.

$$\therefore\ x_{\prime\prime} = x + y \cos C,$$

$$y_{\prime\prime} = y \sin C;$$

$$\therefore\ \tan\phi = -\frac{dx_{\prime\prime}}{dy_{\prime\prime}} = -\frac{1 + \dfrac{dy}{dx}\cos C}{\dfrac{dy}{dx}\sin C};$$

but $\qquad xy = c^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1);$

$$\therefore\ \frac{dy}{dx} = -\frac{c^2}{x^2};$$

$$\therefore\ \tan\phi = \frac{x^2 - c^2\cos C}{c^2 \sin C} \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Also, if $Cg$ be taken $= \frac{2}{3}CN$, $g$ will be the center of gravity of the triangle $PCQ$. The co-ordinates of this point, taken parallel to $AC$ and $BC$, are therefore $\frac{2}{3}x$ and $\frac{2}{3}y$. Let $G$ be the center of gravity of the figure. The line $Gg$ will therefore be vertical, or perpendicular to $PQ$, when the body is in its position of equilibrium; and hence, calling $x_{\prime}$ and $y_{\prime}$ the co-ordinates of $G$, and observing that the inclination of $Gg$ to $AC$ is represented by $\phi$, we have for the equation to $Gg$,

$$y_{\prime} - \frac{2}{3}y = \frac{\sin\phi}{\sin(C-\phi)}\left(x_{\prime} - \frac{2}{3}x\right),$$

$$\text{or} \quad y_{,} - \frac{2}{3}y = \frac{\tan\phi}{\sin C - \tan\phi \,.\, \cos C}\left(x_{,} - \frac{2}{3}x\right) \ldots\ldots (3);$$

eliminating ($y$) and ($\tan\phi$) between the equations (1), (2) and (3),

$$y_{,} - \frac{2}{3}\frac{c^2}{x} = \frac{x^2 - c^2\cos C}{c^2 - x^2\cos C}\left(x_{,} - \frac{2}{3}x\right);$$

whence, by reduction, we obtain

$$x^4 - \frac{3}{2}(x_{,} + y_{,}\cos C)x^3 + \frac{3}{2}(y_{,} + x_{,}\cos C)\,c^2x - c^4 = 0 \ldots\ldots (4).$$

The value of $x$, obtained from this equation, being substituted in (2), the value of $\phi$ or the inclination of the side $AC$ to the vertical is known.

If the line joining $C$ and $G$ bisect the angle $ACB$, drawing $GH$ parallel to $BC$, it is clear that

$$CH = HG, \quad \text{or} \quad x_{,} = y_{,}.$$

The general equation becomes, therefore,

$$x^4 - \frac{3}{2}x_{,}(1 + \cos C)\,x^3 + \frac{3}{2}x_{,}(1 + \cos C)\,c^2x - c^4 = 0,$$

$$\text{or} \quad x^4 - c^4 - 3x_{,} \,.\, x \,.\, (x^2 - c^2) \,.\, \cos^2\frac{C}{2} = 0;$$

$$\left.\begin{aligned} \therefore\ x^2 - c^2 &= 0, \\ \text{and} \quad x^2 + c^2 - 3x_{,} \,.\, x\cos^2\frac{C}{2} &= 0 \end{aligned}\right\}.$$

The first equation gives, by substitution in equation (2),

$$\phi = \frac{C}{2}.$$

Whence it appears, that the figure is in equilibrium when $CG$ is in a vertical position.

From the second equation we obtain

$$x = \frac{3}{2} x_{,} \cos^2 \frac{C}{2} \pm \sqrt{\frac{9}{4} x_{,}^2 \cos^4 \frac{C}{2} - c^2}.$$

If we draw $GI$ at right angles to $CA$, and represent $CI$ by $h$, we shall have

$$h = \overline{CG} \cos \frac{C}{2} = 2x_{,} \cos^2 \frac{C}{2};$$

$$\therefore \; x = \frac{3}{4} h \pm \sqrt{\frac{9}{16} h^2 - c^2};$$

or, substituting for $c$ its value,

$$x = \frac{3}{4} h \pm \sqrt{\frac{9}{16} h^2 - \frac{A\sigma}{2 \sin C}}.$$

Ex. Suppose the figure a sector of a circle:

Here $$CG = \frac{4}{3} a \,.\, \frac{\sin \frac{C}{2}}{C};$$

$$\therefore \; h = \frac{4}{3} a \,.\, \frac{\sin \frac{C}{2} \cos \frac{C}{2}}{C} = \frac{2}{3} a \frac{\sin C}{C}.$$

Also, $$A = \frac{1}{2} a^2 C;$$

$$\therefore \; x = \frac{1}{2} a \frac{\sin C}{C} \pm \sqrt{\frac{1}{4} a^2 \frac{\sin^2 C}{C^2} - \frac{1}{4} a^2 \frac{C\sigma}{\sin C}}$$

$$= \frac{1}{2} a \frac{\sin C}{C} \left\{ 1 \pm \sqrt{1 - \left(\frac{C}{\sin C}\right)^3 \sigma} \right\}.$$

An oblique position of equilibrium is therefore impossible, unless

$$\left(\frac{C}{\sin C}\right)^3 \sigma < 1, \quad \text{or } \sigma < \left(\frac{\sin C}{C}\right)^3.$$

67. If the figure be a rectangle, *one angle* of which only is immersed. Making $C = \frac{\pi}{2}$ in the general equation (4), and taking $x_{,}$ and $y_{,}$ respectively equal to the halves of the sides $a$ and $b$ of the rectangle, we have

$$x^4 - \frac{3}{4}ax^3 + \frac{3}{4}bc^2x - c^4 = 0;$$

from which equation the value of $\phi$ is determined by substitution in (2).

This equation must have at least one positive and one negative root. It may have three positive and one negative root. There cannot, therefore, be more than three positions of equilibrium with one angle immersed.

68. If the figure $AKBC$ be inverted, so that the part not immersed may be in every position a triangle, the conditions of equilibrium are determined by substituting $(1-\sigma)$ for $(\sigma)$ in the general equation (4). For the first condition of equilibrium is satisfied when

$$\text{Area } PQBKA = A \cdot \sigma,$$

$$\text{or when } A - PQC = A \cdot \sigma,$$

$$\text{or when } PQC = A(1-\sigma),$$

which is the same with the first condition in the former case, $(1-\sigma)$ being substituted for $\sigma$.

Also, the second condition of equilibrium is *the same* in both cases. For if $g'$ be the center of gravity of the part $PKBQ$, it is clear that the center of gravity $G$ of the whole will be in the line $gg'$; $g'G$ is therefore perpendicular to $PQ$ when $Gg$ is perpendicular to $PQ$.

69. To determine the conditions of the equilibrium of an area, the lower part of which is a rectangular parallelogram, *two angles* of which are immersed.

$PQ$ (Fig. 22.) the line of flotation; $M$ the center of gravity of the whole area $ACDB$ $(=A)$, and $m$ that of $AQ$, the part immersed:

$$AB = 2a, \quad AP = \lambda, \quad BQ = \lambda'.$$

Draw $MN$ and $mn$ perpendiculars on $AC$, and join $Mm$. Suppose the body in a position of equilibrium, then is $Mm$ perpendicular to $PQ$.

Now,

$$\overline{AQ} \,.\, \overline{mn} = \overline{AV} \,.\, \frac{1}{2}\overline{AB} + \overline{PVQ} \,.\, \frac{2}{3}\overline{PV},$$

$$= 2a^2\lambda + \frac{4}{3}a^2(\lambda' - \lambda),$$

$$= \frac{2}{3}a^2(2\lambda' + \lambda).$$

Also, $$\overline{AQ} \,.\, \overline{An} = \overline{AV} \,.\, \frac{1}{2}\overline{BV} + \overline{PVQ} \,.\, \overline{BV + \frac{1}{3}VQ},$$

$$= a\lambda^2 + \frac{1}{3}a(\lambda' - \lambda)(\lambda' + 2\lambda),$$

$$= \frac{a}{3}(\lambda'^2 + \lambda\lambda' + \lambda^2),$$

$$= \frac{a}{3}\left(\frac{\lambda'^3 - \lambda^3}{\lambda' - \lambda}\right).$$

Now, $AQ = A\sigma$;

$$\therefore \; \overline{mn} = \frac{2a^2}{3A\sigma}(2\lambda' + \lambda), \qquad An = \frac{a}{3A\sigma}\left(\frac{\lambda'^3 - \lambda^3}{\lambda' - \lambda}\right).$$

Let the angle $AKM$, which $AC$ makes with the vertical, be represented by $\theta$;

$$\therefore \; \lambda' - \lambda = 2a \tan VPQ = 2a \tan\theta;$$

$$\text{also} \quad a(\lambda' + \lambda) = AQ = A \,.\, \sigma.$$

$$\text{Let} \quad \frac{A\sigma}{a} = 2c;$$

$$\therefore \lambda' = c + a\tan\theta,$$
$$\lambda = c - a\tan\theta;$$

$$\therefore mn = \frac{a}{3c}(3c + a\tan\theta);$$

$$\text{and} \quad An = \frac{1}{6c}(3c^2 + a^2\tan^2\theta).$$

$$\text{Let } AN = x_{,} \qquad MN = y_{,};$$

$$\therefore \tan\theta = \tan mM\mu = \frac{mn - MN}{AN - An};$$

$$\therefore \tan\theta = \frac{\frac{a}{3c}(3c + a\tan\theta) - y_{,}}{x_{,} - \frac{1}{6c}(3c^2 + a^2\tan^2\theta)};$$

whence by reduction we obtain;

$$\therefore \tan^3\theta + \left\{\frac{3c^2}{a^2} - \frac{6cx_{,}}{a^2} + 2\right\}\tan\theta + \frac{6c(a - y_{,})}{a^2} = 0$$

If a line drawn through $M$ parallel to $AC$, bisect $AB$ $a = y_{,}$;

$$\therefore \tan^3\theta + \left\{\frac{3c^2}{a^2} - \frac{6cx_{,}}{a^2} + 2\right\}\tan\theta = 0;$$

$$\therefore \tan\theta = 0;$$

$$\tan\theta = \pm\sqrt{\frac{6cx_{,}}{a^2} - \frac{3c^2}{a^2} - 2}.$$

If the whole figure be a rectangle, $c = b\sigma$ and $x_{,} = \frac{1}{2}b$;

$$\text{therefore,}\ \tan\theta = \pm\sqrt{\frac{3b^2\sigma}{a^2}(1-\sigma) - 2}.$$

The equation $\tan\theta = 0$, indicates a vertical position. The last equation gives an oblique position with the following conditions. 1st, That the radical be not imaginary. 2nd, That the corresponding values of $\lambda$ and $\lambda'$ be positive. And 3rd, That they be less than $b$.

The first condition gives

$$\frac{3b^2\sigma}{a^2}(1-\sigma) > = 2,$$

$$\text{or,}\ \sigma^2 - \sigma = < -\frac{2}{3}\frac{a^2}{b^2};$$

$$\text{or,}\ \sigma \sim \tfrac{1}{2} < = \sqrt{\frac{1}{4} - \frac{2}{3}\frac{a^2}{b^2}}.$$

The second gives,

$$b\sigma - a\ \tan\ \theta > 0;$$

$$\text{or,}\ b^2\sigma^2 > 3b^2\sigma(1-\sigma) - 2a^2;$$

$$\text{or,}\ \sigma \sim \frac{3}{8} > \sqrt{\frac{9}{64} - \frac{a^2}{2b^2}}.$$

The third condition gives by a similar process,

$$\sigma \sim \frac{5}{8} > = \sqrt{\frac{9}{64} - \frac{a^2}{2b^2}}.$$

It is clear that the third condition is involved in the second.

In the case of the square $b = 2a$, we have therefore for the conditions of oblique equilibrium,

$$\sigma \sim \frac{1}{2} < = \sqrt{\frac{1}{12}},$$

$$\sigma \sim \frac{3}{8} > \frac{1}{8}.$$

Now, $\sigma$ does not lie between $\frac{3}{8}$ and $\frac{1}{2}$, since in that case $\sigma \sim \frac{3}{8}$ would be less than $\frac{1}{2} - \frac{3}{8}$ or $\frac{1}{8}$; contrary to the second condition. $\sigma$ is therefore greater than $\frac{1}{2}$ or less than $\frac{3}{8}$.

$$\text{It is} < = \frac{1}{2} + \sqrt{\frac{1}{12}}, \text{ or, } > = \frac{1}{2} - \sqrt{\frac{1}{12}}.$$

70. To determine the angle $Mmk$, (Fig. 20.) at which the line $Mm$ is inclined to the vertical in any given position of the figure;

$$\tan MSK = \tan mM\mu = \frac{\frac{a}{3c}(3c + a \tan\theta) - y_{,}}{x_{,} - \frac{1}{6c}(3c^2 + a^2 \tan^2\theta)}$$

$$= \frac{6c(a - y_{,}) + 2a^2 \tan\theta}{6cx_{,} - 3c^2 - a^2 \tan^2\theta};$$

$$\tan Mmk = \tan(\theta - MSK)$$

$$= \frac{\tan\theta - \frac{6c(a - y_{,}) + 2a^2\tan\theta}{6cx_{,} - 3c^2 - a^2\tan^2\theta}}{1 + \frac{6c(a - y_{,})\tan\theta + 2a^2\tan^2\theta}{6cx_{,} - 3c^2 - a^2\tan^2\theta}}$$

$$= \frac{-a^2\tan^3\theta + (6cx_{,} - 3c^2 - 2a^2)\tan\theta - 6c(a - y_{,})}{a^2\tan^2\theta + 6c(a - y_{,})\tan\theta + 6cx_{,} - 3c^2},$$

when the figure is symmetrical $a = y_{,}$;

$$\therefore \tan Mmk = \tan\theta \cdot \frac{-a^2\tan^2\theta + 6cx_{,} - 3c^2 - 2a^2}{a^2\tan^2\theta + 6cx_{,} - 3c^2}.$$

If $x_{,} = \frac{1}{2}c$,

$$\tan Mmk = -\frac{\tan^2\theta + 2}{\tan\theta} = -(\tan\theta + 2\cot\theta).$$

As the body revolves through its successive positions of equilibrium, the angle $Mmk$ vanishes. Between each two successive positions of equilibrium, there is, therefore, some position in which this angle is a maximum. In the last case given above, the angle $Mmk$ is a maximum when $\tan\theta = \sqrt{2}$.

71. To determine the force $Pg$ which applied horizontally and in the plane of the figure at the given point $p$, will produce an equilibrium at a given inclination $\theta$ of the axis $M\mu$ to the vertical.

Let $\alpha$ and $\beta$ be the co-ordinates of $p$ from $M$ on the axis $M\mu$, $qp$ the direction of the force on $p$;

$$\therefore\ P \times \overline{Mq} = D\,.\,A\,.\,\overline{Mk},$$

$D$ being the density of the fluid.

Now, $Mq = \alpha\cos\theta + \beta\sin\theta$;

$$\text{also, } \overline{Mm}^2 = (mn - MN)^2 + (AN - An)^2$$

$$= \frac{\{6c(a - y_{,}) + 2a^2\tan\theta\}^2 + \{3c(2x_{,} - c) - a^2\tan^2\theta\}^2}{36c^2}$$

$$\sin^2 Mmk = \{1 + \cot^2 Mmk\}^{-1}.$$

$$= 1 + \left\{\frac{a^2\tan^2\theta + 6c(a - y_{,})\tan\theta + 3c(2x_{,} - c)}{-a^2\tan^3\theta + (6cx_{,} - 3c^2 - 2a^2)\tan\theta - 6c(a - y_{,})}\right\}^2.$$

Now, $\overline{Mk} = \overline{Mm}\sin\overline{Mmk}$;

$$\therefore\ P = \frac{A\,.\,D}{6c(\alpha\cos\theta + \beta\sin\theta)}$$

$$\cdot\left\{\frac{\{6c(a-y_{,}) + 2a^2\tan\theta\}^2 + \{3c(2x_{,} - c) - a^2\tan^2\theta\}^2}{1 + \left\{\frac{a^2\tan^2\theta + 6c(a-y_{,})\tan\theta + 3c(2x_{,} - c)}{-a^2\tan^3\theta + (6cx_{,} - 3c^2 - 2a^2)\tan\theta + 6c(a-y_{,})}\right\}^2}\right\}^{\frac{1}{2}}$$

If the body be symmetrical, or $a = y_{\prime}$

$$P = \frac{A \,.\, D}{6c(a\cos\theta + \beta\sin\theta)} \cdot \left\{ \frac{4a^4\tan^2\theta + \left\{3c(2x_{\prime} - c) - a^2\tan^2\theta\right\}^2}{1 + \cot^2\theta \left\{ \dfrac{a^2\tan^2\theta + 3c(2x_{\prime} - c)}{-a^2\tan^2\theta + 6cx_{\prime} - 3c^2 - 2a^2} \right\}^2} \right\}^{\frac{1}{2}}$$

When $\theta$ is exceeding small,

$$P = \frac{A \,.\, D}{6ca}(6cx_{\prime} - 3c^2 - 2a^2)\,\theta.$$

The force requisite to cause different areas taken as above, to revolve through the same exceeding small angle $\theta$, varies therefore, as

$$\frac{A \,.\, D}{6ca}(6cx_{\prime} - 3c^2 - 2a^2),$$

which quantity may be taken as a measure of their stability.

The maximum value of $P$ represents the force necessary to retain the body in that position in which the point $m$ is most distant from the vertical, through $M$. Any additional force, therefore, or the velocity communicated by the continual action of this through any previous period, will cause it to revolve beyond that position; and (since the distance of the point ($m$) from the vertical afterwards diminishes), *à fortiori* through every succeeding position until passing through its next position of equilibrium, the point $m$ falls on the opposite side of the vertical, and the force $P$ is *augmented* by the pressure of the fluid. The motion of the body is therefore continued, until it attains its next position of equilibrium in an inverted position. The maximum value of $P$, is therefore, that force which is requisite to overturn the body.

72. To find the positions of equilibrium of a cone.

Let $mn$ (Fig. 31.) be the plane of flotation when the cone is in its position of equilibrium. Take $CG$ equal to

$\frac{3}{4}$ths of the axis $CD$. Draw $Ck$ to the center of the plane of flotation $mn$, and take $Cg$ equal to $\frac{3}{4}$ths of $Ck$. Then are $G$ and $g$ the centers of gravity of the cones $ACB$ and $mCn$, and $Gg$ is perpendicular to $mn$. Also, $Dk$ is parallel to $Gg$, ($CG$ and $Cg$ being respectively $\frac{3}{4}$ths of $CD$ and $Ck$); therefore $Dk$ is perpendicular to $mn$. And

$$mk = kn; \qquad \therefore\ Dm = Dn.$$

Let $AD = BD = c,\quad AC = CB = a,\quad ACB = \beta,$

$$Cm = x, \quad Cn = y;$$

$$\therefore\ CD^2 = a^2 - \frac{c^2}{4}, \text{ and } CD \cos\frac{\beta}{2} = \frac{CD^2}{AC} = \frac{4a^2 - c^2}{4a}.$$

Now, $\quad Dm = Dn$;

$$\therefore\ x^2 - 2x\,.\,\overline{CD}\,.\cos\frac{\beta}{2} = y^2 - 2y\,.\,\overline{CD}\,.\cos\frac{\beta}{2};$$

$$\therefore\ x^2 - y^2 - 2(x-y)\,.\,\overline{CD}\,.\cos\frac{\beta}{2} = 0;$$

$$\therefore\ x - y = 0, \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$\text{and } x + y - 2\,.\,\overline{CD}\,.\cos\frac{\beta}{2} = 0;$$

$$\therefore\ x + y = \frac{4a^2 - c^2}{2a} \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

$$\text{Volume of cone } Cmn = \frac{\pi y^3\,.\,\sin^2\beta}{12\cos^2\frac{\beta}{2}}\,.\left(\frac{x}{y}\right)^{\frac{3}{2}} = \frac{\pi}{3}\cos\frac{\beta}{2}\sin^2\frac{\beta}{2}\,(xy)^{\frac{3}{2}},$$

$$\text{and volume of } ACB = \frac{\pi}{3}\sin^2\frac{\beta}{2}\cos\frac{\beta}{2}\,.\,a^3;$$

$$\therefore\ (xy)^{\frac{3}{2}} = a^3\sigma;$$

$$\therefore\ xy = a^2\sigma^{\frac{2}{3}} \ldots\ldots\ldots\ldots\ldots\ldots (3).$$

Therefore, from the first of the equations,

$$x = y = a\,\sigma^{\frac{1}{3}}.$$

From the second and third,

$$x = \frac{(4a^2 - c^2) \pm \sqrt{(4a^2 - c^2)^2 - 16a^4\sigma^{\frac{2}{3}}}}{4a},$$

$$y = \frac{(4a^2 - c^2) \pm \sqrt{(4a^2 - c^2)^2 - 16a^4\sigma^{\frac{2}{3}}}}{4a}.$$

## CHAP. VI.

### ON THE STABILITY OF FLOATING BODIES.

73. If a body floating in equilibrium, and displacing therefore a quantity of fluid of the same weight with itself, and having its center of gravity in the same vertical line with that of the part of it immersed, be elevated or depressed through any given space, and at the same time made to revolve about its center of gravity through any given angle, so that its weight may be other than that of the fluid it displaces, and the center of gravity of the part immersed without the vertical line through its center of gravity: the equilibrium will manifestly be destroyed. The question of stability consists in determining whether the body, when left to itself under these circumstances, will continually recede further from its position of equilibrium, or oscillate about and eventually recover it.

We shall in the following investigation confine ourselves to the case in which the first condition of equilibrium may be supposed to be satisfied in every position the body is made to assume, or its weight constantly to equal that of the fluid it displaces.

Def. The plane of flotation is the section of a floating body made by the horizontal surface of the fluid on which it floats.

74. If the floating body be homogeneous, the center of gravity of the whole is above that of the fluid displaced, which in this case coincides with that of the portion of it immersed. If the density of the lower portion be greater than that of the higher, the center of gravity of the body may lie below that of the fluid displaced. In fact, it is evident that the center of gravity of the body may always be brought below that of the fluid displaced, by diminishing the density of its superior portion and increasing that of the inferior.

75. If there be a plane, with regard to which the parts of a body are symmetrical, any portion of it cut off by another plane perpendicular to the former will be symmetrical with regard to the same plane. Whence it follows, that, if a body thus symmetrical with regard to a certain plane, be partially immersed in a fluid, that plane being vertical; the part immersed (or cut off by the horizontal surface of the fluid) will also be symmetrical with regard to that plane; and the center of gravity of the body itself and (in every position) of the part immersed will lie in it. For as many such planes, therefore, as can be taken in a body, there is at least one position of equilibrium.

76. Let $AVB$ (Fig. 23.) be the section of a floating body made by a vertical plane, about which it is symmetrical; and let motion be communicated to it in a direction parallel to that plane.

Now, the rotatory motion of the body about its center of gravity is the same as though that point were at rest, and the same forces applied.

Conceive the center of gravity $M$ of the body to be fixed; and let $n$ be the center of gravity of the part immersed, and $MK$ and $nk$ verticals through $M$ and $n$. Now, when

the body is in a position of equilibrium, the point $n$ is in the vertical $MK$. Conceive it to revolve out of such a position about the point $M$. Then it is clear, that if the point $n$ move in the *same direction* with the body in respect to $MK$, the tendency of the pressure of the fluid on $n$ will be to continue its revolution; but if the motion of $n$ be with respect to $MK$ in a direction *contrary* to that of the revolution of the body, its tendency will be to check the revolution about $M$, and bring the body back to its former position of equilibrium.

If, therefore, a force be made so to act upon the body as to move it ever so slightly from its position of equilibrium, and then to cease: in the case we have first mentioned, the motion about $M$ will be continued and continually accelerated, and in the other retarded and eventually destroyed. In the one case, the position out of which the body has revolved is said to be a position of unstable, and in the other of stable equilibrium.

77. If, whilst the disturbance is exceeding slight, the point $n$ remain at rest, or move wholly in the vertical $MK$, it is clear that the pressure of the fluid will have no tendency either to restore the body to its position of equilibrium, or to cause it to revolve farther from it. The equilibrium is, in this case, said to be one of indifference. The disturbance being, however, continued, this case will resolve itself into one of the former, or into a fourth case, in which the motion of the point $n$ is in the *same* direction from the vertical, whatever be the direction of the revolution of the body. It is clear that in this last case the body will still farther recede from its position of equilibrium, or reinstate itself according as the disturbance is in the *given* direction of the motion of $n$, or in the opposite direction; and thus in one direction the equilibrium will be stable, and in the other unstable. The equilibrium may in this case be said to be of mixed stability.

If the revolution be in either of the above cases continued, the distance of the point $n$ from $MK$ will at length have

attained a maximum value; after passing through which it will diminish, and ultimately again vanish*, when the center of gravity of the body and of the part immersed being again in the same vertical, there is a second position of equilibrium. The revolution being still farther continued in the same direction; if zero be not a minimum value of the function expressing the distance $mn$, the point $n$ will pass to the opposite side of the vertical, and $mn$ will again increase negatively with regard to its former value until the function has attained its minimum, when $n$ will again approach the vertical and cross it, the body passing through its third position of equilibrium.

If zero be a minimum value of $mn$, the motion of $n$ will be to the same side of the vertical as before, or the position is one of mixed equilibrium.

78. As the body passes through two successive positions, *in which the point n crosses the vertical*, the motion of that point is in opposite directions, and is therefore alternately in the same direction with the revolution of the body, and in the contrary direction; or the positions are one of them unstable and the other stable.

If the second position be one of mixed equilibrium, the point $n$ will *not cross* the vertical. *If* therefore, in the succeeding position *it cross* it, it will cross it in the same direction as it would have done immediately from its first position. The *first* and *third* positions are, therefore, in this case, one of them stable and the other unstable; and generally the positions immediately preceding and following one or any number of mixed equilibriums, are of alternate stability. Hence it follows, that *the positions of stable and unstable equilibrium occur alternately*, although the converse of this

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* It is clear that the value of $mn$ will (under all circumstances) vanish when the body has revolved back into its first position. The curve, which is the locus of $n$, either cuts the vertical therefore, or touches it.

proposition, viz. that *the positions of equilibrium are alternately stable and unstable**, cannot be affirmed.

79. Let $P'Q'$ and $PQ$ be respectively the positions of the plane of flotation when the body is in equilibrium, and when it has been made to revolve about its center of gravity through a small angle $M\mu n\,(=\theta)$. Now,

$$\text{solid } PVQ = \text{solid } P'VQ';$$

therefore, taking away the common part $PO'Q'V$,

$$POO'P' = QOO'Q'.$$

The plane $P'Q'$ having been horizontal, and therefore perpendicular to the plane $P'VQ$; also, the motion having taken place wholly in a direction parallel to this last plane, it follows that the plane $P'Q'$ still remains perpendicular to it. The planes $PQ$ and $P'Q'$, and their common intersection $OO'$, are therefore perpendicular to $P'VQ$.

Now, if we take $M$ to represent the constant volume of the part immersed, $M.\overline{mn}$ represents the momentum of

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* This law of stability is that commonly given in works on Hydrostatics. The following proof is supposed to establish it.

Let the body be made to revolve out of one position of stable equilibrium into another. Now, immediately adjacent to its first position, its tendency is to return to it, or to check the revolution; and immediately adjacent to its second position, its tendency is *from* the first position, or to continue the revolution: and this tendency to revolution clearly passes through every degree of magnitude; there is, therefore, some intermediate position, in passing through which it changes its direction and vanishes: and this position is one of unstable equilibrium. But it does not follow that between the two first positions there is not some *other* position, in passing through which the tendency to revolution vanishes *without changing its direction*. Such a position is clearly one of mixed equilibrium.

The above proof establishes the law given in the text, and no more.

$PVQ$ about a plane $zy$ perpendicular to the plane $PVQ$, and therefore parallel to $OO'$;

$$\therefore\ M\,.\,\overline{mn} = \text{mom}^{\text{m}}.\ PO'Q'V + \text{mom}^{\text{m}}.\ QO'Q'.$$

Now, let $N$ be the center of gravity of the solid $P'VQ'$;

$$\therefore\ M\,.\,\overline{mn_{,}} = \text{mom}^{\text{m}}.\ PO'Q'V + \text{mom}^{\text{m}}.\ PO'P';$$

$$\therefore\ M\,.\,\overline{nn_{,}} = \text{mom}^{\text{m}}.\ QO'Q' - \text{mom}^{\text{m}}.\ PO'P'.$$

Now, since the solids $QOQ'$ and $POP'$ are equal, the difference of their momenta with regard to a vertical plane in a given position, will not vary, if we suppose that plane to move parallel to itself into any other given position; for it is clear that their momenta (being represented by their masses multiplied by the distances of their centers of gravity from that plane) will both be increased or diminished by the same quantity in such motion. Hence, therefore, $M\,.\,\overline{n_{,}n}$ equals the difference of the momenta of $POP'$ and $QOQ'$ about a vertical plane through $OO'$.

Let $ad'$ be an element of $QO'Q'$ contained by planes perpendicular to $PVQ$, and parallel to it; then is the solid content of $ad'$ represented by

$$\overline{ac'} \times \overline{ab} = \frac{1}{2}(ac + a'c')\,\overline{aa'}\,.\,\overline{ab},$$

$$= \frac{1}{2}(ac + a'c')\,\overline{ab'},$$

$$= \frac{1}{2}(ac + ac')\,\overline{cd'}\,.\,\sin\theta\,\cos\theta;$$

therefore, momentum of $ad'$ about a vertical plane through $OO'$,

$$= \frac{1}{4}(ac + ac')^2\,.\,\overline{cd'}\,.\,\sin\theta\,\cos^2\theta,$$

$$= (\text{mom}^{\text{m}}.\ \text{of inertia of plane } cd')\,.\,\sin\theta\,\cos^2\theta;$$

therefore, momentum $QOQ'$

$$= (\text{mom}^{\text{m}}.\text{ of inertia of plane } Q'OO') \,.\, \sin\theta \cos^2\theta.$$

Similarly,

$$\text{mom}^{\text{m}}.\ POP' = -(\text{mom}^{\text{m}}.\text{ of inertia of plane } P'OO') \,.\, \sin\theta\cos^2\theta;$$

$$\therefore\ M\overline{n_{,}n} = (\text{mom}^{\text{m}}.\text{ of inertia of plane } P'Q') \,.\, \sin\theta\cos^2\theta.$$

Also,

$$\text{solid } ad' = \frac{1}{2}(ac + ac') \,.\, \overline{cd'} \,.\, \sin\theta\cos\theta,$$

$$= (\text{mom}^{\text{m}}.\text{ of } cd' \text{ about } OO') \,.\, \sin\theta\cos\theta;$$

therefore, whole solid $QOQ' = (\text{mom}^{\text{m}}.\ Q'OO') \,.\, \sin\theta \,.\, \cos\theta$.

Similarly,

$$\text{solid } POP' = -(\text{mom}^{\text{m}}.\ P'OO') \,.\, \sin\theta \,.\, \cos\theta;$$

$$\therefore\ (\text{mom}^{\text{m}}.\ Q'OO')\sin\theta\cos\theta = -(\text{mom}^{\text{m}}.\ P'OO') \,.\, \sin\theta \,.\, \cos\theta;$$

$$\therefore\ \text{mom}^{\text{m}}.\ Q'OO' + \text{mom}^{\text{m}}.\ P'OO' = 0;$$

therefore $OO'$ passes through the center of gravity of the plane $P'Q'$.

Let now $I$ represent the momentum of inertia of the plane $P'Q'$ about the axis $OO'$, thus passing through its center of gravity.

$$\therefore\ \overline{n_{,}n} = \frac{I}{M}\sin\theta \,.\, \cos^2\theta;$$

$$\therefore\ N\mu = \frac{I}{M}\cos^2\theta.$$

And if $MN$ be represented by $a$,

$$M\mu = \frac{I}{M}\cos^2\theta \mp a,$$

the sign $\mp$ being taken according as the center of gravity of the body lies above or below that of the part of it immersed.

Since $\theta$ is exceeding small, $\cos\theta = 1$;

$$\therefore\ M\mu_{,} = \frac{I}{M} \mp a.$$

Now, if the point $n$ move to that side of $MK$ towards which the body is revolving, it is clear that $\mu$ will lie below $M$, or $M\mu$ be negative; and if to the other side, positive. The equilibrium is therefore stable or unstable, according as $\frac{I}{M} \mp a$ is positive or negative. If $M\mu = 0$, the equilibrium belongs to the ambiguous case which is said to be indifferent, but may in fact be stable, unstable, or mixed.

80. If the part of the body immersed be a solid of revolution, and the center of gravity of the whole lie in its axis, it will be cut symmetrically by any vertical plane through that point; and in whatever direction the disturbance takes place, the conditions we have assumed will be satisfied.

Now, in this case, the plane of flotation will be a circle, and the axis of rotation $OO'$ its diameter. If, therefore, the radius of the plane of flotation be represented by $(a)$, then

$$I = \frac{1}{4}\pi a^4 \quad \text{and} \quad M\mu = \frac{\pi a^4}{4M} \mp a.$$

Ex. To determine the stability of a cone when floating vertically.

Let $a$ and $a_{,}$ be the radii of the base and plane of flotation respectively, and $b$ and $b_{,}$ their distances from the vertex. And first, suppose the cone to float with its vertex downwards. The distances of the centers of gravity of the whole, and the part immersed from the vertex, are then respectively

$$\frac{3}{4}b \quad \text{and} \quad \frac{3}{4}b_{,}; \qquad \therefore\ a = \frac{3}{4}(b - b_{,}).$$

Also, the whole and part immersed being similar cones, are to one another as the cubes of the radii of their bases, or of their altitudes;

$$\therefore\ a_{,}^3 = \sigma a^3, \qquad b_{,}^3 = \sigma b^3.$$

Now, $$M = \frac{1}{3}\pi a_{,}^{2} b_{,},$$

and the negative sign is to be taken, since the center of gravity of the body clearly lies above that of the part immersed;

$$\therefore \; M\mu = \frac{\pi a_{,}^{4}}{\frac{4}{3}\pi a_{,}^{2} b_{,}} - \frac{3}{4}(b - b_{,})$$

$$= \frac{3}{4}\frac{a_{,}^{2}}{b_{,}} - \frac{3}{4}(b - b_{,}),$$

$$= \frac{3}{4}\left\{\frac{a^2\sigma^{\frac{1}{3}}}{b} - b\,(1 - \sigma^{\frac{1}{3}})\right\}.$$

Suppose the cone to float with its base downwards. We have them as before, since the whole cone and the portion above the surface of the fluid are similar solids;

$$a_{,}^{3} = a^3(1 - \sigma), \qquad b_{,}^{3} = b^3(1 - \sigma).$$

Also, the distance of the center of gravity of the frustrum immersed, from the vertex

$$= \frac{3}{4}\frac{b^4 - b_{,}^{4}}{b^3 - b_{,}^{3}};$$

$$\therefore \; a = \frac{3}{4}\frac{b^4 - b_{,}^{4}}{b^3 - b_{,}^{3}} - \frac{3}{4}b,$$

$$a = \frac{3}{4}\frac{b_{,}^{3}(b - b_{,})}{b^3 - b_{,}^{3}} = \frac{3}{4}\frac{b^3(1 - \sigma)\,\{b - b\,(1 - \sigma)^{\frac{1}{3}}\}}{b^3\sigma};$$

$$\therefore \; M\mu = \frac{\pi a^4(1 - \sigma)^{\frac{4}{3}}}{\frac{4}{3}\pi a^2 b\sigma} - \frac{3}{4}\left(\frac{1 - \sigma}{\sigma}\right)\left\{1 - (1 - \sigma)^{\frac{1}{3}}\right\}.b$$

$$= \frac{3}{4}\left(\frac{1 - \sigma}{\sigma}\right)\left\{\left(\frac{a^2}{b} + b\right)(1 - \sigma)^{\frac{1}{3}} - b\right\}.$$

Suppose $a = b$; therefore, in the first case,

$$M\mu = \frac{3}{4}a\left\{2\sigma^{\frac{1}{3}} - 1\right\}.$$

The equilibrium is therefore stable, indifferent, or unstable, according as

$$\sigma > = < \frac{1}{8}.$$

In the second case,

$$M\mu = \frac{3}{4}a\left(\frac{1-\sigma}{\sigma}\right)\left\{2(1-\sigma)^{\frac{1}{3}} - 1\right\},$$

and the stability is dependant on the conditions,

$$\sigma < = > \frac{7}{8}.$$

From the above it appears that if in the second position the equilibrium be stable, it is unstable in the opposite position.

Ex. 2. Let the body be a paraboloid immersed with its vertex downwards, axis $= a$, radius of base $= b$; and let $a_{,}$ and $b_{,}$ be similarly taken with regard to the part immersed;

$$\therefore\ a = \frac{2}{3}(b - b_{,}),$$

$$M = \frac{1}{2}\pi a_{,}^{2} b_{,} = \frac{1}{2}\pi a^{2} b\sigma;$$

$$\text{also, } \frac{a_{,}^{2}}{b_{,}} = \frac{a^{2}}{b} = \text{parameter} = 2c;$$

$$\therefore\ b_{,}^{2} = b^{2}\sigma,$$

$$M\mu = \frac{\pi a_{,}^{4}}{2\pi a_{,}^{2} b_{,}} - \frac{2}{3}(b - b\sqrt{\sigma}),$$

$$= c - \frac{2}{3}b(1 - \sqrt{\sigma}).$$

The stability is therefore determined by the conditions

$$\sigma > = < \left(1 - \frac{3c}{2b}\right)^2.$$

81. If the body or the part of it immersed, be generated by the motion of a plane perpendicular to itself, and be so immersed that the generating plane may be in a vertical position, the plane of flotation will be a rectangular parallelogram; and it is clear that the conditions supposed, will be satisfied for a motion parallel to either of its sides*. Let these be represented by $b$ and $c$, and let the motion take place parallel to the side $b$;

$$\therefore\ I = \frac{cb^3}{12},$$

$$\text{and } M\mu = \frac{cb^3}{12M} \mp a.$$

Ex. Let it be required to find the stability of a rectangular parallelopipedon floating vertically on a fluid. The direction of disturbance being parallel to one of its sides.

Take $a$, $b$, $c$ to represent the edges of the parallelopipedon, ($a$) being vertical. Let $a_{\prime}$ represent the depth to which the body sinks, and $\frac{1}{2}h$ the distance of its center of gravity from its base.

Then, $$a = \tfrac{1}{2}h - \tfrac{1}{2}a_{\prime},$$

$$\text{and } M = a_{\prime}bc = abc\sigma;$$

$$\therefore\ M\mu = \frac{cb^3}{12abc\sigma} - \tfrac{1}{2}(h - a_{\prime}),$$

$$\text{and } a_{\prime} = a\sigma;$$

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* Provided only the center of gravity of the whole be in the plane about which the part immersed is symmetrical.

$$\therefore\ M\mu = \frac{b^2}{12a\sigma} - \frac{1}{2}(h - a\sigma,)$$

and the stability is determined by the conditions,

$$h < = > \frac{b^2}{6a\sigma} + a\sigma.$$

If the body be of uniform density, $h = a$, the conditions of equilibrium are, therefore,

$$a < = > \frac{b^2}{6a\sigma} + a\sigma,$$

$$\text{or,}\quad \left(\frac{b}{a}\right) > = < \sqrt{6\sigma(1 - \sigma)}.$$

For the cube, we have

$$\sigma \sim \frac{1}{2} > = < \sqrt{\frac{1}{12}}.$$

The conditions which have been shewn to obtain in the vertical position of the parallelopipedon, apply in a converse order to the oblique positions of equilibrium on each side of it.

Generally, where the body is symmetrical, the oblique positions immediately adjacent on each side, to its vertical position of equilibrium, are clearly similar and of the same stability. In cases where such oblique positions are possible*, the intervening vertical position is therefore *not* one of mixed stability, since the positions immediately preceding and following such a position, are of *opposite* stability.

It has been shewn that the equilibrium of a cube in an oblique position is impossible, unless

$$\sigma \sim \frac{1}{2} = < \sqrt{\frac{1}{12}}.$$

Now, these are precisely the conditions, which obtaining, the vertical position is unstable or indifferent; if therefore, the

* Unless they be both of indifferent stability.

vertical position of the cube be not one of indifference, the oblique positions are stable.

We have hitherto supposed the body to be symmetrical about a certain plane, that plane to be vertical when the body is in its position of equilibrium, and the motion communicated to be such as to cause it to remain vertical during the body's revolution. On which hypothesis the motion of the center of gravity of the part immersed takes place wholly within it.

82. Let us now take the more general case, in which the motion of the point $(n)$ is otherwise than in a vertical plane through $M$.

Let $P'VQ$ (Fig. 23.) be that vertical section of the body through its center of gravity $M$, which is perpendicular to the intersection $OO'$ of the planes of flotation $PQ$ and $P'Q'$. The plane $P'VQ$ is therefore perpendicular to $P'Q'$. But $MN$ is perpendicular to $P'Q'$; therefore, $MN$ is in the plane $P'VQ$. Let $n$ be the *projection* of the center of gravity of the part immersed on this plane, and through $n$ draw the vertical $n\mu$.

Now it is clear, that the tendency of the pressure of the fluid will be to check the rotation about the axis $My$, or to continue it; that is, to diminish the angle $\theta$, made by the planes $PQ$ and $P'Q'$, or increase it according as the center of gravity of the part immersed, or its projection $n$ lies in the same direction from the plane $zy$, with the revolution of the body, or in the opposite direction. When left to itself, therefore, the motion of the body will be to diminish the angle $\theta$, or to increase it, according as $M\mu$ is positive, or negative.

Now precisely as before, it may be shewn that

$$M\mu = \frac{I}{M} \mp a,$$

where $I$ is the momentum of inertia of the plane $P'Q'$ about the axis $OO'$. If $N$ and $N'$ be respectively the momenta about

the principal axes of the plane $P'Q'$, and $\psi$ the angle made by $OO'$ with the axis of *greatest* momentum, then

$$I = N\cos^2\psi + N'\sin^2\psi;$$

$$\therefore\ M\mu = \frac{N\cos^2\psi + N'\sin^2\psi}{M} \mp a.$$

83. If this quantity be essentially positive, whatever be the value of $\psi$, it follows, that in every position which the body can assume when left to itself, the tendency of the pressure of the fluid will be to check its further motion; it will therefore again be brought to rest, and its equilibrium is *stable*. And similarly, if $M\mu$ be negative for every value of $\psi$, the pressure, into whatever position the body may be brought, will be to continue its motion, or the equilibrium is *unstable*. But if for certain values of $\psi$, $M\mu$ be negative, and for other values positive, there are certain directions of disturbance in which the equilibrium is stable, and certain others in which it is unstable: and the body, when left to itself, may eventually pass into a position in which the sign of $M\mu$ is changed, so that its motion may at first be to return to its position of equilibrium, and then, before it attains that position, to recede again from it, and conversely. The equilibrium is, therefore, in this case, *doubtful*.

Now it is clear, that if $M\mu$ or $\left(\frac{I}{M} \mp a\right)$ be positive for the minimum value of $I$, it is positive for every other value: and if it be negative for the maximum value, it is negative for every other value.

We, in fact, therefore determine whether a body be stable or unstable under every possible circumstance, by considering its stability with respect to motion, perpendicular to the principal axes of its plane of flotation. It is stable in every direction, if stable with regard to that axis about which the momentum is a minimum: it is unstable in every direction, if unstable with regard to that about which it is a maximum.

If the equilibrium be doubtful, an axis may be taken, for which $M\mu$ vanishes, or *about which* the equilibrium may be mixed. To determine this axis, we shall have

$$\frac{N\cos^2\psi + N'\sin^2\psi}{M} - a = 0;$$

$$\therefore \sin\psi = \sqrt{\frac{N - Ma}{N - N'}}.$$

Thus, in the case in which the body is generated by the motion of a plane perpendicular to itself,

$$\sin\psi = \sqrt{\frac{cb^3 - 12Ma}{cb(b^2 - c^2)}}.$$

84. If $\mu$ and $\mu'$ be the values of $M\mu$ in reference to the principal axes of the plane of flotation, we shall have

$$\frac{N'}{M} \mp a = \mu,$$

$$\frac{N'}{M} \mp a = \mu';$$

$$\therefore \frac{N\cos^2\psi + N'\sin^2\psi}{M} \mp a = \mu\cos^2\psi + \mu'\sin^2\psi;$$

therefore, generally,

$$M\mu = \mu\cos^2\psi + \mu'\sin^2\psi.$$

From which it is clear (as above) that, if $\mu$ and $\mu'$ have the *same sign*, the equilibrium is in every direction stable or unstable, according as that sign is positive or negative: and that if they have *different signs*, there are certain directions in which it is stable, and others in which it is unstable.

Ex. 1. To determine generally the stability of a rectangular parallelopipedon.

Here the principal axes of the plane of flotation are clearly parallel to the sides;

$$\therefore\ \mu = \frac{b^2}{12a\sigma} - \frac{1}{2}(h - a\sigma),$$

$$\mu' = \frac{c^2}{12a\sigma} - \frac{1}{2}(h - a\sigma);$$

$$\therefore\ M\mu = \frac{b^2\cos^2\psi + c^2\sin^2\psi}{12a\sigma} - \frac{1}{2}(h - a\sigma).$$

If the horizontal section of the figure be a square,

$$b = c, \quad \text{and} \quad M\mu = \frac{b^2}{12a\sigma} - \frac{1}{2}(h - a\sigma);$$

or the stability is the same, whatever be the direction of the disturbance.

Ex. 2. To determine the stability of an ellipsoid.

Calling $a$, $b$, $c$ the axes (of which $c$ is vertical), and $x$ the distance of the plane of flotation from the centre, we have (see Poisson, *Mech.* Art. 115.)

$$a = \frac{3}{4}(c + x)\left(\frac{c^2 - x^2}{2c^2 - cx - x^2}\right),$$

$$= \frac{3}{4}\,\frac{(c + x)^2}{2c + x}.$$

Also, the semi-axes of the plane of flotation are

$$\frac{b}{c}(c^2 - x^2)^{\frac{1}{2}}, \quad \text{and} \quad \frac{a}{c}(c^2 - x^2)^{\frac{1}{2}};$$

$$\therefore\ N' = \frac{1}{4}\pi \cdot \frac{b^3}{c^3}(c^2 - x^2)^{\frac{3}{2}} \cdot \frac{a}{c} \cdot (c^2 - x^2)^{\frac{1}{2}},$$

$$= \frac{1}{4}\,\frac{\pi a b^3}{c^4}(c^2 - x^2)^2.$$

Also, $M = \frac{4}{3}\pi abc\sigma$;

$$\therefore\ \mu' = \frac{\frac{1}{4}\pi\frac{ab^3}{c^4}(c^2-x^2)^2}{\frac{4}{3}\pi abc\sigma} - \frac{3}{4}\frac{(c+x)^2}{(2c+x)}$$

$$= \frac{3}{16}\frac{b^2}{c^5\sigma}(c^2-x^2)^2 - \frac{3}{4}\frac{(c+x)^2}{(2c+x)}.$$

Similarly,

$$\mu = \frac{3}{16}\frac{a^2}{c^5\sigma}(c^2-x^2)^2 - \frac{3}{4}\frac{(c+x)^2}{(2c+x)};$$

$$\therefore\ M\mu = \frac{3(c^2-x^2)^2}{16c^3\sigma}(a^2\cos^2\psi + b^2\sin^2\psi) - \frac{3}{4}\frac{(c+x)^2}{(2c+x)}.$$

The value of $x$ is determined by the equation

$$x^3 - 3xc^2 + 2c^3(1-2\sigma) = 0.$$

If $\sigma = \frac{1}{2}$, $x = 0$, or the plane of flotation coincides with a principal section of the ellipse;

$$\therefore\ M\mu = \frac{3(a^2\cos^2\psi + b^2\sin^2\psi)}{8c} - \frac{3}{8}c;$$

$$\therefore\ \mu = \frac{3}{8c}(a^2-c^2), \qquad \mu' = \frac{3}{8c}(b^2-c^2).$$

If, therefore, $c$ be the least of the three axes, the equilibrium is stable in every direction. If it be less than one axis and greater than the other, it is doubtful.

# CHAP. VII.

## ON THE OSCILLATIONS OF FLOATING BODIES.

85. The motion of the center of gravity of a body or system of bodies, is the same as though the whole mass were collected in it, and the forces applied to the whole were made to act directly upon it.

Now, the forces impressed upon a floating body are its weight and the pressure of the fluid, both of which are in a vertical direction. If, therefore, it be moved from its position of equilibrium through any space and then left to itself, no velocity having first been communicated to it, the motion of its center of gravity will be wholly in a vertical direction.

Suppose the body $PVQ$ (Fig. 23'.) after having revolved through an angle $\theta$ about its center of gravity $M$ to descend until its plane of flotation is $P''Q''$.

Let $S$ equal the volume of the solid, $\zeta$ the descent $P''p$ of the center of gravity, and $D$, $D'$ the densities of the solid and fluid; therefore the impressed moving force

$$= SDg - \overline{P''VQ''} . D' . g$$

$$= S . D . g - \overline{M + PQ''} . D'g$$

$$= - \overline{PQ''} . D' . g;$$

also the momentum of the effective forces is represented by

$$S . D . \frac{d^2\zeta}{dt^2};$$

$$\therefore S . D . \frac{d^2\zeta}{dt^2} = - \overline{PQ''} . D' . g;$$

$$\therefore \frac{d^2\zeta}{dt^2} = - \frac{g . \overline{PQ''}}{M}.$$

Now, if the oscillation be exceeding small, calling $K$ the area of the plane of flotation, $P''Q''$ or $P'Q'$.

$$PQ'' = K\zeta;$$

$$\therefore \frac{d^2\zeta}{dt^2} = - \frac{g . K}{M}\zeta;$$

$$\therefore \frac{2d\zeta d^2\zeta}{dt^2} = - \frac{2gK}{M}\zeta d\zeta;$$

$$\therefore \left(\frac{d\zeta}{dt}\right)^2 = \frac{gK}{M}(a^2 - \zeta^2),$$

($a$) being the height of the center of gravity above its position of equilibrium at the beginning of the motion, or its depth below it;

$$\therefore t = \left(\frac{M}{Kg}\right)^{\frac{1}{2}} \int \frac{-d\zeta}{\sqrt{a^2 - \zeta^2}};$$

$$\therefore t = \left(\frac{M}{Kg}\right)^{\frac{1}{2}} \cos^{-1}\left(\frac{\zeta}{a}\right);$$

$$\therefore \zeta = a . \cos\left(\frac{Kg}{M}\right)^{\frac{1}{2}} . t.$$

When $\zeta = 0$,

$$\left(\frac{Kg}{M}\right)^{\frac{1}{2}} . t = \frac{\pi}{2}.$$

It appears then that the time of the body's passing from its greatest distance to its position of equilibrium, is

$$\frac{1}{2}\left(\frac{M}{Kg}\right)^{\frac{1}{2}} . \pi,$$

and this expression being independent of ($a$) it follows, that, whatever the greatest distance may be, the time is the same; and that the oscillations are isochronous. Also that the time of an oscillation, or the time during which a body continues to ascend or descend, is represented by

$$\left(\frac{M}{Kg}\right)^{\frac{1}{2}} . \pi.$$

Ex. Suppose the body a paraboloid of revolution. Let $a$ be its axis, and $b$ the radius of its base; also let $y$ be the radius of the plane of flotation;

$$\therefore\ S = \tfrac{1}{2}\pi a b^2;$$

$$\therefore\ \tfrac{1}{2}\pi y^2 x = \tfrac{1}{2}\pi b^2 a\sigma;$$

$$\text{also,}\quad \frac{y^2}{x} = \frac{b^2}{a};$$

$$\therefore\ y^4 = b^4\sigma;$$

$$\therefore\ K = \pi b^2\sqrt{\sigma};$$

therefore, time of vertical oscillation

$$= \pi\sqrt{\frac{\frac{1}{2}\pi a b^2\sigma}{\pi b^2 . \sqrt{\sigma} . g}},$$

$$= \pi\left(\frac{a}{2g}\right)^{\frac{1}{2}} . \sigma^{\frac{1}{4}}.$$

86. Suppose the oscillation to be finite, and let the motion of the body be wholly in a vertical direction;

$$\therefore\ PQ'' = \int K d\zeta;$$

$$\therefore\ \frac{d^2\zeta}{dt^2} = -g\int\frac{K d\zeta}{M}.$$

Ex. 1. Let the body be a cylinder. Then $K$ is constant:

$$\therefore\ \frac{d^2\zeta}{dt^2} = -\frac{gK\zeta}{M};$$

whence proceeding exactly as in the case of small oscillations, we find that a cylinder, to whatever depth it may be plunged, if left to itself, will oscillate isochronously, and that its constant time of oscillation will be

$$\pi\sqrt{\frac{M}{Kg}} = \pi\sqrt{\frac{l\sigma}{g}},$$

calling $l$ the length of the cylinder.

Ex. 2. Suppose the body a cone with its vertex downwards. Let $c$ be its altitude, and $b$ the radius of its base. Calling $x$ the distance of the plane of flotation from the vertex of the cone at any period of the motion, and $x_{,}$ the value of $x$, when the body is in its position of equilibrium,

$$PQ' = \int K d\zeta = \frac{1}{3}\pi\left(\frac{b^2}{c^2}x^3 - \frac{b^2}{c^2}x_{,}^3\right);$$

$$\therefore \frac{d^2\zeta}{dt^2} = -\frac{\frac{1}{3}\pi\frac{b^2}{c^2}(x^3 - x_{,}^3)}{\frac{1}{3}\pi b^2 c\sigma}.$$

Now, $x = x_{,} + \zeta$;

$$\therefore \frac{d^2x}{dt^2} = \frac{d^2\zeta}{dt^2};$$

$$\therefore \frac{d^2x}{dt^2} = -\frac{x^3 - x_{,}^3}{c^3\sigma};$$

$$\therefore \left(\frac{dx}{dt}\right)^2 = 2\frac{\frac{1}{4}(a^4 - x^4) - x_{,}^3(a-x)}{c^3\sigma};$$

$a$ being the value of $x$ at the beginning of the motion.

Now,

$$\tfrac{1}{3}\pi\frac{b^2}{c^2}x_{,}^3 = \tfrac{1}{3}\pi b^2 c\sigma;$$

$$\therefore x_{,}^3 = c^3\sigma;$$

$$\therefore \left(\frac{dx}{dt}\right)^2 = \frac{(a^4 - x^4) - 4c^3\sigma(a-x)}{2c^3\sigma}.$$

The equation

$$(a^4 - x^4) - 4c^3\sigma(a - x) = 0,$$

determines the value of $x$, for which the velocity is nothing, or the extent of the oscillation. It is manifestly satisfied by taking $x = a$ according to the hypothesis, or

$$x^3 + ax^2 + a^2x + a^3 - 4c^3\sigma = 0.$$

### *On the Oscillations of a Floating Body about its Center of Gravity.*

87. The motion of rotation of a body about an axis passing through its center of gravity, is the same as though that center were fixed and the same forces were applied.

Suppose the point $M$ (Fig. 23.) to be at rest. And to take the simplest case, let the motion be parallel to a vertical plane, about which the body is symmetrical, so that the motion of the center of gravity $n$ of the part immersed may be wholly in that plane.

Now, $M$ being the center of gravity of the body, the momentum of rotation produced in it (about that point) by its own weight is nothing; and the whole momentum of the impressed forces about $M$ arises from the pressure of the fluid, and is represented by

$$MD' . \overline{Mn''} . g,$$

or by

$$MD' . g \left\{ \frac{I}{M} \mp a \right\} \theta *.$$

---

* Since the momentum of rotation varies as $\theta$, the oscillation observes the same laws with that of the pendulum, we may therefore at once conclude that it is isochronous, and that its duration is

$$\frac{k\pi}{\sqrt{\left(\frac{I}{M} \mp a\right) g}}.$$

Also the momentum of rotation of the effective forces about a horizontal axis through $M$, is represented by

$$\frac{d^2\theta}{dt^2} \cdot Sk^2 D,$$

$Sk^2$ being the momentum of inertia of the whole solid. Whence, by D'Alembert's principle, we obtain

$$-\frac{d^2\theta}{dt^2} S \cdot k^2 \cdot D = M \cdot D' \cdot g \left\{\frac{I}{M} \mp a\right\} \theta.$$

The negative sign is taken, the position being supposed of stable equilibrium, and therefore the tendency of the force always to diminish the angle $\theta$;

$$\therefore \left(\frac{d\theta}{dt}\right)^2 = g \left(\frac{\frac{I}{M} \mp a}{k^2}\right) (\Theta - \theta^2);$$

$\Theta$ being the amplitude of the oscillation. Calling ($t$) the time, measured from the instant of greatest amplitude,

$$t = \frac{k}{\sqrt{\left(\frac{I}{M} \mp a\right) g}} \cos^{-1}\left(\frac{\theta}{\Theta}\right) \ldots\ldots\ldots\ldots\ldots (1).$$

Since, when $t = 0$, $\theta = \Theta$, and therefore $\cos^{-1}\left(\frac{\theta}{\Theta}\right) = 0$. For the value of $t$, when $\theta = 0$, we have

$$\frac{k}{\sqrt{\left(\frac{I}{M} \mp a\right) g}} \tfrac{1}{2} \pi.$$

And this expression being independent of $\Theta$, it appears that the oscillations are performed in the same time, whatever be their amplitude; and that the whole time of each oscillation is

$$\frac{k\pi}{\sqrt{\left(\frac{I}{M} \mp a\right) g}}.$$

When $\frac{I}{M} \mp a$ is negative, the integral (1) becomes a logarithmic function; and $\theta$ continually increases with $t$, or as we have shown before, the equilibrium is unstable.

Ex. Find the time of a small oscillation of a cylinder floating vertically.

Let $x$ be the distance of any transverse section of the cylinder from its center of gravity; and let $a$ be the radius of its base and $b$ its height. Now the momentum of inertia of the section about its diameter is $\frac{1}{4}\pi a^4$. About an axis parallel to its diameter, therefore, and through the center of gravity of the cylinder, at distance $x$, its momentum is $\frac{1}{4}\pi a^4 + \pi a^2 x^2$. And hence, passing to the whole momentum about this last axis, we have $Sk^2 = \int(\frac{1}{4}\pi a^4 + \pi a^2 x^2)\,dx$. And taking the integral from $x = +\frac{1}{2}b$ to $x = -\frac{1}{2}b$,

$$Sk^2 = \pi a^2 \left\{\frac{1}{4}a^2 b + \frac{1}{12}b^3\right\};$$

$$\therefore\ k^2 = \frac{1}{4}a^2 + \frac{1}{12}b^2.$$

Also, $$I = \frac{1}{4}\pi a^4;$$

$$\therefore\ \frac{I}{M} = \frac{a^2}{4b\sigma}.$$

And the depth to which the cylinder is immersed when in its position of equilibrium, is $b\sigma$;

$$\therefore\ a = \frac{1}{2}b(1-\sigma);$$

$$\therefore\ t = \pi\sqrt{\frac{\frac{1}{4}a^2 + \frac{1}{12}b^2}{\left\{\frac{a^2}{4b\sigma} - \frac{1}{2}b(1-\sigma)\right\}g}},$$

$$t = \pi\sqrt{\frac{(3a^2 + b^2)\,b\sigma}{\{3a^2 - 6b^2\sigma(1-\sigma)\}\,g}}.$$

88. Let us now proceed, more generally, to consider the oscillations of a floating body, whatever may be the direction of the disturbance.

Let us suppose the volume of the part immersed to remain constant during the revolution of the body. In its motion from its position of equilibrium, let the body describe the angles $X$, $Y$, $Z$, about the rectangular axes $Mx$, $My$, $Mz$, passing through its center of gravity $M$, of which let the axis $Mz$ be vertical.

Let $PQ$ (Fig. 23.) be the position of the plane of flotation after the body has revolved about the axis $My$ through the angle $Y$, and $PO''QO'''$ (Fig. 23′.) that after it has revolved about the axis $Mx$ through the angle $X$.

Now it is clear, that if in any position of the body a motion be communicated about the axis of $z$, the part immersed will remain unaltered, and therefore the impressed force and effective motion of the body in describing the angles $X$ and $Y$, will be the same as though no such motion had existed.

Let us suppose the motion to have been wholly about the axes $x$ and $y$, so that $PO''QVO'''$ (Fig. 23′.) may be the final position of the body.

By the revolution about the axis $My$, the part immersed is increased by the sector $QO'Q'$, and diminished by the sector $PO'P'$; and in the revolution about $Mx$, it is farther increased by $PO''QO$, and diminished by $PO'''QO'$. Therefore, on the whole, the part immersed equals

$$P'VQ' + QOQ' - POP' + OPO'' - O'PO'''.$$

Now, the axes $OO'$ (Fig. 23.) and $PQ$ (Fig. 23′.) are clearly parallel to $My$ and $Mx$ respectively; the difference of the momenta of the *equal* sectors $POP'$ and $QOQ'$ about the plane $zy$, is therefore represented (Art. 81.) by $I \sin Y \cos^2 Y$;

and the difference of the momenta of $OPO''$ and $O'PO'''$ about $zx$ by $I'$ sin $X$ cos ${}^2X$; $I$ and $I'$ representing respectively the momenta of inertia of the planes $P'OQ'O'$ and $POQO'$ about the axes $OO'$ and $PQ$. Or if $X$ and $Y$ be exceeding small, these quantities are represented by $IY$ and $I'X$, where $I$ and $I'$ are the momenta of inertia of the plane $P'Q'$ about its axes $OO'$ and $P'Q'$.

Let $dK$ represent any element of the plane $P'Q'$, and let $k$ be the distance of the axis $OO'$ from the plane $zy$; then momentum $POO'P'$ — momentum $QOO'Q'$ about plane $zx = \sin Y \cos Y \int (k-x)\, y dK$. Let $\int (k-x)\, y dK$ be represented by $N'$, and let $N$ be similarly taken with regard to the plane $zy$ and the solids $O'PO'''$, $OPO''$. Then momentum $OPO''Q - O'PO'''Q$ about $zy = N \sin X \cos X$.

Now the momentum of the impressed force about $zy$

$$= \text{mom}^{\text{m}}.\ P'VQ' + \text{mom}^{\text{m}}.\ QOQ' - \text{mom}^{\text{m}}.\ POP'$$
$$+ \text{mom}^{\text{m}}.\ OPO'' - \text{mom}^{\text{m}}.\ O'PO'''$$
$$= \{aYM + IY + NX\}\ Dg.$$

Similarly, the momentum of the impressed force about $zx$,

$$= \{aXM + I'X + N'Y\}\ Dg.$$

Now let $x_{,}\ y_{,}\ z_{,}$ be the co-ordinates of any element $dm$ of the body.

Then by the general equations for the motion of a solid body about its center of gravity, we have

$$\int \frac{x_{,} d^2 y_{,} - y_{,} d^2 x_{,}}{dt^2}\, dm = 0,$$

$$\int \frac{y d^2 z_{,} - z_{,} d^2 y_{,}}{dt^2}\, dm = \{(aM + I)\ Y + NX\}\ \sigma g,$$

$$\int \frac{y_{,} d^2 z_{,} - z_{,} d^2 y_{,}}{dt^2}\, dm = \{(aM + I')\ X + N'Y\}\ \sigma g.$$

Let $x\ y\ z$ have been the co-ordinates of the point $(x_, y_, z_)$ or the element $dm$, when the body was in its position of equilibrium. To determine the values of $x_,\ y_,\ z_,$ in terms $x\ y\ z$, suppose the body to revolve about the axis $Oz$ (Fig. 13.) through an angle $ROQ=Z$. It is clear that the value of $z$ will remain unaltered.

Let $x_{,,}\ y_{,,}$ be the new values of $x$ and $y$,

$$\therefore x_{,,}=OR \,.\, \cos RON = OQ \,.\, \cos (QOM+Z),$$

$$=OQ \,.\, \cos QOM \,.\, \cos Z - OQ \sin QOM \sin Z,$$

$$\left.\begin{aligned} &= x \cos Z - y \sin Z \\ y_{,,} &= OQ \sin (QOM+Z) \\ &= y \cos Z + x \sin Z \end{aligned}\right\} \dots\dots\dots\dots\dots (1.)$$

Let the body *now* be supposed to revolve about the axis $Ox$ through the angle $X$, and let $z_{,,}$ and $y_,$ be the corresponding values of $z$ and $y_{,,}$. Then, as before,

$$\left.\begin{aligned} z_{,,} &= z \cos X - y_{,,} \sin X \\ y_, &= y_{,,} \cos X + z \sin X \end{aligned}\right\} \dots\dots\dots\dots\dots (2.)$$

Let the body *now*, finally, revolve about $Oy$, $z_{,,}$ and $x_{,,}$ becoming $z_,$ and $x_,$,

$$\left.\begin{aligned} z_, &= z_{,,} \cos Y - x_{,,} \sin Y \\ x_, &= x_{,,} \cos Y + z_{,,} \sin Y \end{aligned}\right\} \dots\dots\dots\dots\dots (3.)$$

Eliminating the values of $x_{,,}\ y_{,,}\ z_{,,}$ between the equations (1), (2), (3), and observing that since $X$, $Y$, $Z$ are exceedingly small, we may consider them as equal to their sines, represent their cosines by unity, and omit terms which involve their product; we obtain

$$x_, = x - yZ + zY,$$

$$y_, = y + xZ + zX,$$

$$z_, = z - yX - xY.$$

Differentiating with respect to the time,

$$\frac{dx_{\prime}}{dt} = -y\,\frac{dZ}{dt} + z\,\frac{dY}{dt},$$

$$\frac{dy_{\prime}}{dt} = \quad x\,\frac{dZ}{dt} + z\,\frac{dX}{dt},$$

$$\frac{dz_{\prime}}{dt} = -y\,\frac{dX}{dt} - x\,\frac{dY}{dt};$$

therefore, omitting the terms involving $X$, $Y$ and $Z$,

$$\frac{x_{\prime}dz_{\prime}}{dt} = -x^2\,\frac{dY}{dt} - xy\,\frac{dX}{dt},$$

$$\frac{z_{\prime}dx_{\prime}}{dt} = +z^2\,\frac{dY}{dt} - zy\,\frac{dZ}{dt},$$

$$\frac{y_{\prime}dz_{\prime}}{dt} = -y^2\,\frac{dX}{dt} - xy\,\frac{dY}{dt},$$

$$\frac{z_{\prime}dy_{\prime}}{dt} = +z^2\,\frac{dX}{dt} + zx\,\frac{dZ}{dt},$$

$$\frac{x_{\prime}dy_{\prime}}{dt} = +x^2\,\frac{dZ}{dt} + xz\,\frac{dX}{dt},$$

$$\frac{y_{\prime}dx_{\prime}}{dt} = -y^2\,\frac{dZ}{dt} + yz\,\frac{dY}{dt};$$

$$\therefore\ \frac{x_{\prime}d^2y_{\prime} - y_{\prime}d^2x_{\prime}}{dt^2} = (x^2+y^2)\,\frac{d^2Z}{dt^2} + xz\,\frac{d^2X}{dt^2} - zy\,\frac{d^2Y}{dt^2},$$

$$\frac{x_{\prime}d^2z_{\prime} - z_{\prime}d^2x_{\prime}}{dt^2} = -\,(x^2+z^2)\,\frac{d^2Y}{dt^2} - xy\,\frac{d^2X}{dt^2} + zy\,\frac{d^2Z}{dt^2},$$

$$\frac{y_{\prime}d^2z_{\prime} - z_{\prime}d^2y_{\prime}}{dt^2} = -\,(y^2+z^2)\,\frac{d^2X}{dt^2} - xy\,\frac{d^2Y}{dt^2} - zx\,\frac{d^2Z}{dt^2};$$

$$\therefore\ \frac{d^2Z}{dt^2}\int(x^2+y^2)\,dm + \frac{d^2X}{dt^2}\int xz\,dm - \frac{d^2Z}{dt^2}\int zy\,dm = 0$$

$$-\,\frac{d^2Y}{dt^2}\int(x^2+z^2)\,dm - \frac{d^2X}{dt^2}\int xy\,dm + \frac{d^2Z}{dt^2}\int zy\,dm$$

$$= \{(aM+I)\;Y + NX\}\,\sigma g,$$

$$-\frac{d^2X}{dt^2}\int(y^2+z^2)\,dm-\frac{d^2Y}{dt^2}\int xy\,dm-\frac{d^2Z}{dt^2}\int zx\,dm$$
$$=\{aM+I')\,X+N'Y\}\,\sigma g.$$

Let the integrals $\int(x^2+y^2)\,dm, \int(x^2+z^2)\,dm, \int(y^2+z^2)\,dm$ be represented by $A, A', A''$ respectively, and $\int zx\,dm, \int zy\,dm, \int xy\,dm$ by $B, B', B''$. Also let $N\sigma g$, $N'\sigma g$, $(aM+I)\,\sigma g$, $(aM+I')\,\sigma g$ be represented by $C, C', D, D'$ respectively,

$$\left.\begin{array}{l} \therefore\ A\dfrac{d^2Z}{dt^2}+B\dfrac{d^2X}{dt^2}-B'\dfrac{d^2Y}{dt^2}=0 \\ A'\dfrac{d^2Y}{dt^2}+B''\dfrac{d^2X}{dt^2}-B'\dfrac{d^2Z}{dt^2}+CX+DY=0 \\ A''\dfrac{d^2X}{dt^2}+B''\dfrac{d^2Y}{dt^2}+B\dfrac{d^2Z}{dt^2}+C'Y+D'X=0 \end{array}\right\}\ldots\ldots(A.)$$

Eliminating $\dfrac{d^2Z}{dt^2}$, we obtain

$$\frac{d^2X}{dt^2}+\frac{AA'-B'^2}{AB''+BB'}\cdot\frac{d^2Y}{dt^2}+\frac{CAX}{AB''+BB'}+\frac{DAY}{AB''+BB'}=0,$$

$$\frac{d^2X}{dt^2}+\frac{AB''+BB'}{AA''-B^2}\,\frac{d^2Y}{dt^2}+\frac{C'AY}{AA''-B^2}+\frac{D'AX}{AA''-B^2}=0.$$

To integrate these equations, let them be added, the first having been multiplied by the indeterminate quantity $\mu$.

$$\left.\begin{array}{l} \therefore\ \dfrac{d^2X}{dt^2}+\dfrac{\mu\dfrac{AA'-B'^2}{AB''+BB'}+\dfrac{AB''+BB'}{AA''-B^2}}{\mu+1}\cdot\dfrac{d^2Y}{dt^2}, \\ +\dfrac{\dfrac{CA\mu}{AB''+BB'}+\dfrac{D'A}{AA''-B^2}}{\mu+1}\cdot X+\dfrac{\dfrac{DA\mu}{AB''+BB'}+\dfrac{CA}{AA''-B^2}}{\mu+1},\ Y \end{array}\right\}=0.$$

Let the two last terms of this equation be identical with $\epsilon\,(X+\lambda Y)$ where $\lambda$ represents the coefficient of the second term;

$$\therefore \lambda = \frac{\mu \dfrac{AA' - B'^2}{AB'' + BB'} + \dfrac{AB'' + BB'}{AA'' - B^2}}{\mu + 1},$$

$$\epsilon = \frac{\dfrac{CA\mu}{AB'' + BB'} + \dfrac{D'A}{AA'' - B^2}}{\mu + 1},$$

$$\epsilon\lambda = \frac{\dfrac{DA\mu}{AB'' + BB'} + \dfrac{CA}{AA'' - B^2}}{\mu + 1}.$$

Let $\lambda$, $\lambda'$, $\epsilon$, $\epsilon'$, be the values of $\lambda$ and $\epsilon$ resulting from these equations.

Assume $\psi = X + \lambda Y$ and $\psi' = X + \lambda' Y$;

$$\therefore \frac{d^2\psi}{dt^2} + \epsilon\psi = 0,$$

$$\frac{d^2\psi'}{dt^2} + \epsilon'\psi' = 0.$$

Let now $\alpha$ $\beta$ $\gamma$ be the *initial* angular velocities about the axes $x$ $y$ $z$; and $\Psi$ $\Psi'$ the corresponding values of $\dfrac{d\psi}{dt}$ and $\dfrac{d\psi'}{dt}$; so that when $\psi$ and $\psi' = 0$, $\dfrac{d\psi}{dt} = \alpha + \lambda\beta = \Psi$, and $\dfrac{d\psi'}{dt} = \alpha + \lambda'\beta = \Psi'$. Therefore integrating the above equations,

$$\left(\frac{d\psi}{dt}\right)^2 + \epsilon\psi^2 = \Psi^2,$$

$$\left(\frac{d\psi'}{dt}\right)^2 + \epsilon'\psi'^2 = \Psi'^2;$$

$$\therefore \psi = X + \lambda Y = \frac{\Psi}{\sqrt{\epsilon}} \text{ vers } t\sqrt{\epsilon},$$

$$\psi' = X + \lambda' Y = \frac{\Psi'}{\sqrt{\epsilon'}} \text{ vers } t\sqrt{\epsilon'};$$

$$\therefore\ X = -\frac{\lambda'.\Psi.\text{vers}\, t\sqrt{\epsilon}}{(\lambda-\lambda')\sqrt{\epsilon}} + \frac{\lambda.\Psi'.\text{vers}\, t\sqrt{\epsilon'}}{(\lambda-\lambda')\sqrt{\epsilon'}},$$

$$Y = \frac{\Psi\ \text{vers}\, t\sqrt{\epsilon}}{(\lambda-\lambda')\sqrt{\epsilon}} - \frac{\Psi'.\text{vers}\, t\sqrt{\epsilon'}}{(\lambda-\lambda')\sqrt{\epsilon'}}.$$

Integrating the first of the equations ($A$), we obtain

$$(AZ + BX - B'Y) = (A\gamma + Ba - B'\beta)\, t;$$

$$\left.\begin{aligned}
\therefore\ X = &-\frac{\lambda'.\Psi}{(\lambda-\lambda')\sqrt{\epsilon}}\text{vers}\frac{AZ+BX-B'Y}{A\gamma+Ba-B'\beta}\sqrt{\epsilon}\\
&+\frac{\lambda.\Psi'.}{(\lambda-\lambda')\sqrt{\epsilon'}}\text{vers}\frac{AZ+BX-B'Y}{A\gamma+Ba-B\beta}\sqrt{\epsilon'},\\
Y = &\frac{\Psi}{(\lambda-\lambda')\sqrt{\epsilon}}.\text{vers}\frac{AZ+BX-B'Y}{A\gamma+Ba-B'\beta}\sqrt{\epsilon}\\
&-\frac{\Psi'}{(\lambda-\lambda')\sqrt{\epsilon'}}\text{vers}\frac{AZ+BX-B'Y}{A\gamma+Ba-B'\beta}\sqrt{\epsilon'}.
\end{aligned}\right\}(B.)$$

By which equations the motion of the body is completely determined.

It is clear, that as long as $\epsilon$ and $\epsilon'$ are *positive* quantities, the corresponding integrals are circular functions, and, therefore, that the values of $X$ and $Y$ for certain values of $t$ perpetually diminish, or that the body, when left to itself, after oscillating about its position of equilibrium eventually returns to it. The equilibrium is, therefore, in this case stable. But if the quantities $\epsilon$ and $\epsilon'$ be, one or both, negative, then the corresponding values of $X$ and $Y$ are both logarithmic functions, and after a certain period increase (positively or negatively)* with the time, or the equilibrium is unstable.

---

* After disturbance, the body may in this case *once* pass through its position of equilibrium, $X$ and $Y$ vanishing. The motion, however, will be continued through it, these quantities afterwards increasing negatively with the time.

Let us take the case in which the axes of $x$ $y$ and $z$ are principal axes of rotation, and in which the principal axes of the plane of flotation $P'Q'$ are in the planes $zx$ and $zy$. A case which manifestly occurs where the body is symmetrical.

Here $B = B' = B'' = 0, \quad C = C' = 0$;

$$\therefore \; A \frac{d^2 Z}{dt^2} = 0,$$

$$A' \frac{d^2 Y}{dt^2} + DY = 0,$$

$$A'' \frac{d^2 X}{dt^2} + D'X = 0;$$

$$\therefore \; \left(\frac{dZ}{dt}\right)^2 = \gamma^2,$$

$$A' \left(\frac{dY}{dt}\right)^2 + DY^2 = A'\beta^2,$$

$$A'' \left(\frac{dX}{dt}\right)^2 + D'X^2 = A''\alpha^2;$$

$$\therefore \; Z = \gamma t,$$

$$Y = \beta \left(\frac{A'}{D}\right)^{\frac{1}{2}} \text{vers} \left(\frac{D}{A'}\right)^{\frac{1}{2}} t,$$

$$X = \alpha \left(\frac{A''}{D'}\right)^{\frac{1}{2}} \text{vers} \left(\frac{D'}{A''}\right)^{\frac{1}{2}} t.$$

The equilibrium is stable or unstable, according as $D$ and $D'$ or $(aM + I)$ and $(aM + I')$ are positive or (one or both) negative. It is clear that the equilibrium can be unstable only in the case in which $a$ is negative, or the center of gravity of the body above that of the part immersed.

The motion of the body is determined by the equations

$$Y = \beta \left(\frac{A'}{D}\right)^{\frac{1}{2}} \text{vers} \left(\frac{D}{A'}\right)^{\frac{1}{2}} \frac{Z}{\gamma},$$

$$\text{and } X = \alpha \left(\frac{A''}{D'}\right)^{\frac{1}{2}} \text{vers} \left(\frac{D'}{A''}\right)^{\frac{1}{2}} \frac{Z}{\gamma}.$$

# CHAP. VIII.

## ON THE EQUILIBRIUM OF VESSELS CONTAINING FLUID.

89. Let the vessel $PAQ$ (Fig. 32.) containing fluid be supported by a horizontal plane with which it is in contact in the point $H$.

Let $M$ be the center of gravity of the vessel, and $N$ that of the contained fluid. Draw the vertical $HR$ and $Mm$, $Nn$ perpendiculars upon it. Now calling $A$ the mass of the vessel, $M$ that of the contained fluid, and $\sigma$ the ratio of their specific gravities, it is clear that since there is an equilibrium,

$$\overline{Nn} \,.\, M = \overline{Mm} \,.\, A \,.\, \sigma.$$

Ex. 1. Conceive the vessel to be generated by the motion of a parabola perpendicular to its plane. Draw $HL$ parallel to the axis $AR$ and $PL$ perpendicular to $HL$. Let the inclination of $AR$ to the vertical $= \theta$, and $SH = r$;

$$\therefore\ M = PAQ = \frac{4}{3}\,\overline{HK}\,.\,\overline{PL}.$$

Also $\overline{PL}^2 = 4r\,.\,\overline{HK}$;

$$\therefore\ M = \frac{8}{3}\sqrt{r}\,.\,\overline{HK}^{\frac{3}{2}}; \qquad \therefore\ HK = \frac{\left(\frac{3}{8}M\right)^{\frac{2}{3}}}{r^{\frac{1}{3}}};$$

$$\therefore\ HN = \frac{3}{5}HK = \frac{3\left(\frac{3}{8}M\right)^{\frac{2}{3}}}{5r^{\frac{1}{3}}}; \qquad \therefore\ Nn = \frac{3\left(\frac{3}{8}M\right)^{\frac{2}{3}}}{5r^{\frac{1}{3}}}\,.\,\sin\theta.$$

Again, $SR = SH = r$. And, if $AM = h$, and $4c$ be the *latus rectum* of the parabola, $SM = h - c$;

$$\therefore\ RM = r + c - h; \qquad \therefore\ Mm = (r + c - h)\sin\theta;$$

$$\therefore\ \frac{3\left(\frac{3}{8}M\right)^{\frac{2}{3}}}{5r^{\frac{1}{3}}} \cdot \sin\theta \cdot M = (r + c - h) \cdot \sin\theta \cdot A \cdot \sigma;$$

$$\therefore\ r^{\frac{4}{3}} + (c - h)\, r^{\frac{1}{3}} - \frac{3}{5}\left(\frac{3}{8}\right)^{\frac{2}{3}} \cdot \frac{M^{\frac{5}{3}}}{A\sigma} = 0.$$

Ex. 2. To find the oblique position of equilibrium of a vessel in the form of a rectangular parallelopipedon containing fluid.

Let $B$ (Fig. 29.) be the angle on which the vessel is supported. Draw the vertical $BM'$, and let $M$ and $m$ be respectively the centers of gravity of the vessel and fluid. Draw $MM'$ and $mm'$ parallel to $AB$;

$$\therefore\ \overline{mm'} \cdot M = \overline{MM'} \cdot A\sigma.$$

Now adopting the notation of (Art. 71.)

$$\overline{mm'} = \overline{nm'} - \overline{nm} = 2a - An\tan\theta - nm.$$

Also (Art. 71.)

$$An = \frac{3c^2 + a^2\tan^2\theta}{6c}, \text{ and } mn = \frac{a}{3c}(3c + a\tan\theta),$$

$$\text{where } c = \frac{M}{a}.$$

Also, if $x_{,}$ and $y_{,}$ be the co-ordinates of $M$,

$$MM' = y_{,} + x_{,}\tan\theta - 2a;$$

$$\therefore\ \left\{2a - \frac{3c^2\tan\theta + a^2\tan^3\theta}{6c} - \frac{a}{3c}(3c + a\tan\theta)\right\} M$$

$$= \{y_{,} + x_{,}\tan\theta - 2a\}\ A\sigma,$$

whence, by reduction,

$$\tan^3\theta + \left(\frac{6Ax_{\prime}\sigma}{a^3} + \frac{3c^2}{a^2} + 2\right)\tan\theta + 6\left(\frac{y_{\prime} - 2a}{a^3}A\sigma - \frac{c}{a}\right) = 0.$$

If the weight of the vessel be neglected, $x_{\prime}$ and $y_{\prime}$ vanish, and we have

$$\tan^3\theta + \left(\frac{3c^2}{a^2} + 2\right)\tan\theta - 6\left(\frac{2A\sigma}{a^2} + \frac{c}{a}\right) = 0.$$

90. The above problems bear a close analogy to that case of the equilibrium of floating bodies in which the body partially rests on the bottom, or is attached to it by a cord, as in the case of a buoy.

Ex. 1. To find the position of equilibrium of a conical buoy whose vertex is retained at a given depth.

Let $ACD$ (Fig. 31.) $=\alpha$, $CM=a$, $MCD=\theta$, $CD=b$;

$$\therefore\ Mm = a\tan(\theta-\alpha),\ Mn = a\tan(\theta+\alpha);$$

$$\therefore\ Mk = \frac{1}{2}a\left\{\tan(\theta-\alpha)+\tan(\theta+\alpha)\right\} = \frac{a\tan\theta\sec^2\alpha}{1-\tan^2\theta\,.\,\sec^2\alpha}.$$

Now if $G$ be the center of gravity of the cone and $g$ that of the part immersed, $Cg = \frac{3}{4}Ck$;

$$\therefore\ M\gamma = \frac{3}{4}Mk = \frac{3}{4}\frac{a\tan\theta\,.\,\sec^2\alpha}{1-\tan^2\theta\,.\,\sec^2\alpha}.$$

Also, $Cm = a\sec(\theta-\alpha)$ and $Cn = a\sec(\theta+\alpha)$;

therefore, content of cone $Cmn$

$$= \frac{\pi}{3}\cos\alpha\,.\,\sin^2\alpha\,.\,a^3\left\{\sec(\theta+\alpha)\sec(\theta-\alpha)\right\}^{\frac{3}{2}},$$

$$= \frac{\pi}{3}a^3\,.\,\frac{\sin^2\alpha\,.\,\cos\alpha}{\left\{\cos^2\theta\,.\,\cos^2\alpha-\sin^2\theta\sin^2\alpha\right\}^{\frac{3}{2}}},$$

$$= \frac{\pi}{3}a^3\,.\,\frac{\tan^2\alpha\,.\,\sec^3\theta}{\left\{1-\tan^2\alpha\,.\,\tan^2\theta\right\}^{\frac{3}{2}}}.$$

Also $CG = \frac{3}{4} b$; $\therefore$ $CN = \frac{3}{4} b \sin \theta$;

$$\text{cone } CAB = \frac{1}{3} \pi b^3 \tan^2 \alpha;$$

$$\therefore \frac{\pi}{4} a^4 . \frac{\tan^2 \alpha \sec^2 \alpha . \sec^4 \theta . \sin \theta}{\{1 - \tan^2 \theta . \tan^2 \alpha\}^{\frac{5}{2}}} = \frac{\pi}{4} b^4 . \sigma . \tan^2 \alpha . \sin \theta;$$

$$\therefore \frac{a^4 \sec^2 \alpha . \sec^4 \theta}{\{1 - \tan^2 \theta . \tan^2 \alpha\}^{\frac{5}{2}}} = b^4 \sigma.$$

Let $AD = c$, and assume $1 - \tan^2 \theta \tan^2 \alpha = x^2$, whence

$$x^5 - \frac{a^4 \sec^2 \alpha}{c^4 \sigma} x^4 + \frac{2a^4 \sec^4 \alpha}{c^4 \sigma} x^2 - \frac{a^4 \sec^6 \alpha}{c^4 \sigma} = 0.$$

*On the Stability of Vessels containing Fluid.*

91. If $H$ (Fig. 23.) be the point in the surface of the vessel in contact with the plane on which it is supported, it is clear that the body when left to itself, will tend to return to its vertical position, be indifferent to motion, or recede further from it, according as

$$\overline{M\gamma} . A\sigma >=< \overline{\gamma\mu} . M.$$

$H\gamma$ being a vertical through $H$, and $n\mu$ through the center of gravity $n$ of the contained fluid.

Now the disturbance being small, $N\mu = \frac{I}{M}$ (Art. 80.), where $I$ is the momentum of inertia of the surface of the fluid about an axis passing through its center of gravity, and perpendicular to the direction of the motion: also $\gamma$ is the center of curvature. Let $V\gamma = \gamma$, $VM = \kappa$, $VN = \kappa'$;

$$\therefore M\gamma = \gamma - \kappa, \quad \gamma\mu = \frac{I}{M} - (\gamma - \kappa');$$

therefore, the equilibrium is determined by the conditions

$$(\gamma-\kappa)\,A\sigma >=< I-(\gamma-\kappa')\,M.$$

Ex. In the paraboloid of revolution, if $4c$ represent the *latus rectum*, and $x_{,}$ the distance of the surface from the vertex;

$$\gamma=2c,\quad \kappa'=\frac{2}{3}x_{,},\ M=2\pi c x_{,}^2,\ I=4\pi c^2 x_{,}^2.$$

The conditions of stability are, therefore,

$$(2c-\kappa)\,A\sigma >=< 4\pi c^2 x_{,}^2-\left(2c-\frac{2}{3}x_{,}\right)2\pi c x_{,}^2,$$

$$\ldots\ldots\ldots\ldots\ >=< \frac{4}{3}\pi c x_{,}^3.$$

92. Suppose the vessel $AP$ (Fig. 34.) to rest upon the *curved* surface $A'P'$. It is required to determine the conditions of stability.

The vessel having revolved slightly from its position of equilibrium, let $Q$ be the point of contact of the two surfaces. Draw the normal $\gamma Q\gamma'$. Then are $\gamma$ and $\gamma'$ the centers of curvature at $A$ and $A'$. Draw the vertical $QK$, and let $M$, $N$, and $\mu$ be taken as before. Now it is clear that the vessel will tend to return to its position of equilibrium, be indifferent to further motion, or tend to continue it, according as

$$\overline{MK}\,.\,A\sigma >=< \overline{\mu K}\,.\,M.$$

Let $AM=\kappa$, $AN=\kappa'$, $A\gamma=\gamma$, $A'\gamma'=\gamma'$.

The conditions of stability are, therefore,

$$(AK-\kappa)\,.\,A\sigma >=< \left(\kappa'+\frac{I}{M}-AK\right)M,$$

$$\text{or } AK >=< \frac{A\sigma\kappa+M\kappa'+I}{A\sigma+M}.$$

Now by $\triangle^{s}$ $\gamma A \gamma'$ and $\gamma' K Q$. Since $A$ and $A'$ may be considered as coinciding, we have

$$AK = \frac{\gamma\gamma'}{\gamma+\gamma'}.$$

The conditions may be reduced, therefore, to

$$\frac{1}{\gamma} + \frac{1}{\gamma'} <=> \frac{A\sigma + M}{A\sigma\kappa + M\kappa' + I}.$$

## CHAP. IX.

### GENERAL EQUATIONS OF THE EQUILIBRIUM OF FLUIDS.

93. To treat the question in its most general form, we shall consider a fluid mass which may be either homogeneous or heterogeneous, compressible or incompressible, and which has all its particles impelled by given accelerating forces. It is proposed to determine the conditions of its equilibrium.

94. Let $PQ$ (Fig. 11.) be a parallelopipedon of the fluid contained by planes parallel to the rectangular co-ordinate planes $xy$, $zx$, $zy$. Let $x$, $y$, $z$, be the co-ordinates, $X$, $Y$, $Z$, the accelerating forces, and $D$ the density at $P$. Also, let $\Delta x$, $\Delta y$, $\Delta z$ represent the edges of the parallelopipedon, and $p$ the pressure at $P$ referred to an unit of surface. Further, let it be supposed that the accelerating forces $X$, $Y$, $Z$, and the density $D$ are the same for every point of the parallelopipedon.

Intersect the mass $PQ$ by a plane $MN$ parallel to either of the co-ordinate planes, as $zy$. Now the increment of pressure generated on $MN$ by the action of the force $X$ on the fluid mass $PM$, is represented by

$$X \,.\, D \,.\, \overline{PN} \,.\, \overline{MN};$$

and this pressure is propagated throughout the fluid $NQ$, and produces, on every point of it, and therefore at $Q$, a pressure which, referred to an unit of surface, is represented by

$$X \cdot D \cdot \overline{PN}.$$

Hence it is manifest, that if $MN$ be made to move up to $QQ'$, the increment of pressure generated at $Q$ by the action of the force $X$ on the whole mass $PQ$, and referred to an unit of surface, is represented by

$$X \cdot D \cdot PQ', \quad \text{or} \quad X \cdot D \cdot \Delta x.$$

Similarly, the increment of pressure generated at $Q$ by the action of the forces $Y$ and $Z$ on the mass $PQ$, are respectively represented by

$$Y \cdot D \cdot \Delta y \quad \text{and} \quad Z \cdot D \cdot \Delta z.$$

Therefore, on the whole, there is generated at $Q$ by the joint action of the forces $X$, $Y$, $Z$ on the solid $PQ$, an increment of pressure, which, referred to an unit of surface, is represented by

$$XD\Delta x + YD\Delta y + ZD\Delta z.$$

On the whole, therefore, the pressure taken on the hypothesis that the accelerating force and the density are the same throughout each element, is represented by

$$\Sigma D\{X\Delta x + Y\Delta y + Z\Delta z\}.$$

Now, according as $\Delta x$, $\Delta y$, $\Delta z$ diminish, does this hypothesis approach to the case which actually obtains, of a continually variable force and density, a limit which it never actually attains for any finite values of these quantities; we have, therefore, accurately,

$$p = \int D\{Xdx + Ydy + Zdz\}^{*} \ldots\ldots\ldots\ldots (\mu).$$

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* The following method of investigating the above equation is exceedingly simple. If $P$ (Fig. 13.) be an elementary plane anywhere situated in a fluid acted upon by forces $X$, $Y$, $Z$ respectively perpendicular to the planes $zy$, $zx$, $xy$. And the fluid being intersected

The integral may be taken with regard to any series of consecutive values of $x$, $y$, $z$. It appears then, that the pressure taken with regard to every such series of values of $x$, $y$, $z$, or in other words, the pressure of every line of fluid particles terminating in the given point, is the same. This principle has been assumed by Newton as a basis to the theory of the equilibrium of fluids. It is clear that the assumption will at once lead to the above equation.

95. Wherever an equilibrium exists, the above equation has been shown to obtain; it is therefore a necessary condition of equilibrium. Now this equation becomes impossible when the right hand member is not an exact differential. It is therefore an essential condition of the equilibrium of a fluid,

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sected transversely by a plane passing through and parallel to $zx$, let the whole of the fluid beyond this plane in the direction $QM$, be supposed to become solid, excepting only the uniform column $QMa$, and which is carried parallel to $yO$, to the surface of the fluid at $a$. The pressure on $P$ will then remain precisely as before. Now, the column $aQ$ exerts on $Q$, by reason of the accelerating force $Y$, a pressure which, when referred to an unit of surface, is represented by $\int Ydy$; and this force being propagated through the fluid, there results from it an equal pressure on $P$. And this is the only (appreciable) pressure generated by the force $Y$ upon $P$. For the whole pressure on $P$ results from the pressure of the fluid column $aQ$, and the fluid lying in the direction of the axis of $y$ from the plane of intersection. Now, this last fluid can, *by reason of the force* $Y$, produce no pressure whatever on $P$, since the direction of the pressure generated in it by $Y$ is directly from that surface. Hence, therefore, it appears that the whole pressure generated by the force $Y$ upon $P$, is that of the fluid column $Qa$, and represented by $\int DYdy$. Similarly, the pressures generated upon $P$ by the forces $X$ and $Z$, are respectively $\int DXdx$ and $\int DYdy$; and therefore, calling the whole pressure upon it, referred to an unit of surface, $p$, we have

$$p = \int DXdx + \int DYdy + \int DZdz$$
$$= \int D\{Xdx + Ydy + Zdz\}.$$

that its density and the forces impressed upon it, should be such as to render the expression

$$D\,(X dx + Y dy + Z dz)$$

an exact differential.

96. From this condition we obtain, by the application of the known rules, (*Lacroix*, *Calc. Integ.* 307.) the following equations, which involve all the conditions necessary to the equilibrium of fluids, as it regards their density and the forces by which they are acted upon;

$$\frac{dDX}{dy} = \frac{dDY}{dx}, \quad \frac{dDX}{dz} = \frac{dDZ}{dx}, \quad \frac{dDY}{dx} = \frac{dDZ}{dy} \ldots (\nu).$$

We may eliminate $D$ by performing the differentiations indicated, multiplying the equations respectively by $Z$, $Y$ and $X$, and adding them together. Whence

$$X\left(\frac{dY}{dz} - \frac{dZ}{dy}\right) + Y\left(\frac{dZ}{dx} - \frac{dX}{dz}\right) + Z\left(\frac{dX}{dy} - \frac{dY}{dx}\right) = 0;$$

an equation which establishes a relation between the forces $X$, $Y$, $Z$ necessary to their producing equilibrium, whatever be the density of the fluid on which they are impressed.

97. Let us now consider a surface taken in the fluid, on every point of which the pressure is the same. It is manifest that since for such a surface $p$ is constant, $dp = 0$; and therefore,

$$Ydx + Ydy + Zdz = 0 \ldots\ldots\ldots\ldots\ldots\ldots (\xi).$$

It is therefore a further condition of the equilibrium of a fluid, that such an arrangement should exist among its parts, as that taking a series of points in it, on all which the pressure is the same, these points should be found in a surface, determined by the above equation. Any number of such surfaces of equal pressure, will manifestly be formed by assigning different values to the arbitrary constant, which enters into the integral of this equation.

98. In all cases of attraction or repulsion directed towards fixed centers of force, in which the intensity is a function of the distance, the quantity $Xdx + Ydy + Zdz$ is an exact differential. (*See Poisson, Mecan. No.* 225.) And in this definition are included, all the forces in nature which can be made to act on the particles of a body at rest. We may, therefore, generally assume

$$Xdx + Ydy + Zdz = d\phi,$$

and, $$dp = D\,.\,d\phi\,\ldots\ldots\ldots\ldots\ldots(o);$$

hence it appears that the product $D\,.\,d\phi$, or the quantity $\frac{dp}{D}$ must, in order that the equilibrium may obtain, be an exact differential. In the former case, $D$ is a function of $\phi$, and may be any function of that quantity; in the latter, it is a function of $(p)$, and therefore also a function of $\phi$.

99. In either case therefore, it appears that an equilibrium cannot exist, unless $p$ and $D$ are both functions of $\phi$, and therefore of one another; and that when $p$ is constant, $D$ is also constant; or that all sufaces of equal pressure, are also of uniform density. Hence, therefore, we gather, that a heterogeneous fluid is disposed when in equilibrio, into a series of homogeneous strata, contained between surfaces, on every portion of each of which the pressure is the same, and whose form is determined by the equation $(\xi)$. The above conditions with regard to the distribution of the parts of a fluid mass, and the variation of its density, include all that is necessary to equilibrium, in such fluids as are presented to us by nature.

100. In the case of incompressible fluids, $Dd\phi$ is an exact differential, and therefore the equilibrium is possible if $D$ be any function whatever of $\phi$. It appears then, that the forces being given, the density of the fluid may be taken to vary, according to an infinity of different laws, with regard to all which the equilibrium will be possible. If the fluid

be elastic, $D$ is exclusively a given function of the pressure, and varies in all known fluids, directly as that pressure; the force being given, it must therefore always vary according to a given law, that the equilibrium may be possible.

101. If $\phi = c$ be the equation to a curved surface, referred to three rectangular co-ordinates, the cosines of the angles which the normal to a point $(x, y, z)$ make with the axis, are respectively

$$\frac{\left(\frac{d\phi}{dx}\right)}{\sqrt{\left(\frac{d\phi}{dx}\right)^2 + \left(\frac{d\phi}{dy}\right)^2 + \left(\frac{d\phi}{dz}\right)^2}}, \quad \frac{\left(\frac{d\phi}{dy}\right)}{\sqrt{\left(\frac{d\phi}{dx}\right)^2 + \left(\frac{d\phi}{dy}\right)^2 + \left(\frac{d\phi}{dz}\right)^2}},$$

$$\frac{\left(\frac{d\phi}{dz}\right)}{\sqrt{\left(\frac{d\phi}{dx}\right)^2 + \left(\frac{d\phi}{dy}\right)^2 + \left(\frac{d\phi}{dz}\right)^2}}.$$

Now, in the surface determined by equation $(\xi)$,

$$\frac{d\phi}{dx}, \frac{d\phi}{dy}, \frac{d\phi}{dz}$$

are respectively equal to $X, Y, Z$; the three quantities given above, are therefore respectively equal to

$$\frac{X}{\sqrt{X^2 + Y^2 + Z^2}}, \frac{Y}{\sqrt{X^2 + Y^2 + Z^2}}, \frac{Z}{\sqrt{X^2 + Y^2 + Z^2}}.$$

Now, these are precisely the cosines of the angles, which the resultant of the forces on the point $(x, y, z)$ makes with the axes. The resultant coincides therefore, in direction with the normal, and at any point of a surface of equal pressure, is perpendicular to that surface.

102. If the extreme surface of a fluid be free, or sustain no external pressure, or if it sustain on every point of it the same pressure, it is manifest that for that surface $dp = 0$;

and that it belongs to the class of what we have defined to be surfaces of equal pressure, and is determined by the equation

$$Xdx + Ydy + Zdz = 0.$$

Ex. Suppose a fluid to be acted upon by forces tending to a fixed center, and equal to $fr$, a function of the distance $r$. Suppose the fluid incompressible and of uniform density, and first let us shew that in this case the equations $(\pi)$ are satisfied, and the equilibrium possible. Resolving the forces on the point $(x, y, z,)$ in the direction of the axes, we have

$$X = -\frac{x}{r}fr, \quad Y = -\frac{y}{r}fr, \quad Z = -\frac{z}{r}fr;$$

$$\begin{aligned}\therefore \frac{dX}{dy} &= -\frac{dfr}{dr}\cdot\frac{dr}{dy}\cdot\frac{x}{r} + fr\cdot\frac{x}{r^2}\cdot\frac{dr}{dy} \\ &= -\frac{dfr}{dr}\cdot\frac{y}{r}\cdot\frac{x}{r} + fr\cdot\frac{x}{r^2}\cdot\frac{y}{r} \\ &= \frac{xy}{r^2}\left\{-\frac{dfr}{dr} + \frac{fr}{r}\right\}.\end{aligned}$$

And similarly,

$$\begin{aligned}\frac{dY}{dx} &= -\frac{dfr}{dr}\cdot\frac{dr}{dx}\cdot\frac{y}{r} + fr\cdot\frac{y}{r^2}\cdot\frac{dr}{dx} \\ &= -\frac{dfr}{dr}\cdot\frac{y}{r}\cdot\frac{x}{r} + fr\cdot\frac{y}{r^2}\cdot\frac{x}{r} \\ &= \frac{xy}{r^2}\left\{-\frac{dfr}{dr} + \frac{fr}{r}\right\};\end{aligned}$$

$$\therefore \frac{dX}{dy} = \frac{dY}{dx},$$

and similarly it may be shown that

$$\frac{dX}{dz} = \frac{dZ}{dx}, \qquad \frac{dY}{dz} = \frac{dZ}{dy},$$

and the density $D$ is constant; therefore it is manifest that the equations $(\pi)$ are satisfied in the case we have assumed.

The same demonstration will apply to the case in which the density is variable, and a function of the distance $r$, if $fr$ be taken to represent the product of the density and accelerating force. It applies also to the case of elastic fluids, since it has been shown that where the density is a function of the pressure, the only condition necessary to equilibrium, is that

$$Xdx + Ydy + Zdz$$

should be an exact differential, which it is proved to be above.

Having shewn that the forces applied to the fluid are such as by a proper distribution of its parts, are sufficient to produce an equilibrium; it remains to consider what that distribution must be. Substituting for $X$, $Y$, $Z$, their values in the equation, ($\nu$) we obtain

$$-\frac{fr}{r}xdx - \frac{fr}{r}ydy - \frac{fr}{r}zdz = 0;$$

$$\therefore \int \frac{fr}{r}\{xdx + ydy + zdz\} = c.$$

$$\int(frdr) = c;$$

$$\therefore\ Fr = \text{constant};$$

$$\therefore\ r = \text{constant}.$$

103. Hence, therefore, it appears that the extreme surface of the fluid, and every surface of equal pressure within it, must at every point be at the same distance from the center. Now each surface of equal pressure, is also of uniform density. It is therefore necessary to the equilibrium, if the fluid be of variable density, that it be disposed in a series of homogeneous spherical layers. If it be incompressible and of uniform density, the above condition reduces itself simply to this, that its external form be spherical.

104. The above reasoning manifestly holds whether we consider the fluid as forming a complete sphere, or being retained in a limited space by the sides of a vessel: in either case the form of its surface will be spherical, having for its

center the center of force. Thus the surface of any portion of fluid at the earth's surface, will be a portion of a spherical surface having its center in the center of the earth; that is, it will be horizontal, or of the same form with the earth's surface. And this will be true, whatever be the form of the vessel in which it is contained.

In the case of gravity, we have, considering it as constant, and taking the axis of $z$ in the vertical,

$$dp = D\{gdz\};$$
$$\therefore p = Dgz + P,$$

taking $z$ from the surface of the fluid to any depth $z$, and calling $P$ the external pressure of the atmosphere. Now, $Dgz$ is the weight of a column of the fluid of the same depth. Hence, therefore, it appears that the pressure at the depth $z$, is the weight of such a column added to the pressure on the surface.

105. Ex. 2. To determine the form of the surface of equal pressure in a fluid which is acted upon by a central accelerating force, varying directly as the distance, and which is further made to revolve with a given uniform velocity, about a fixed axis passing through its center of force.

The central force at the distance unity being $m$, the forces on the point $(x, y, z,)$ resolved in the directions of the axes, are $-mx$, $-my$, $-mz$. Also, the axis of rotation being taken for the axis of $z$, the force generated by the rotatory motion in the direction of that axis, will be nothing. But in a plane perpendicular to it, if the point $(x, y, z)$ be at a distance $\rho$ from it, and $a$ the angular velocity, there will be generated a centrifugal force represented by

$$\frac{(\text{vel.})^2}{\rho} = \frac{(a\rho)^2}{\rho} = a^2\rho,$$

which resolved in the directions of $x$ and $y$, becomes $a^2x$, $a^2y$, both forces being positive, since they tend to increase the co-ordinates. We have, therefore, on the whole,

$$X = -mx + a^2x, \quad Y = -my + a^2y, \quad Z = -mz.$$

by which it is manifest that the equations $(\pi)$ are satisfied. For the surface of equal pressure,

$$(a^2 - m)\,x\,dx + (a^2 - m)\,y\,dy - mz\,dz = 0;$$

$$\therefore\ (a^2 - m)\,x^2 + (a^2 - m)\,y^2 - mz^2 = c,$$

the equation to a prolate spheroid, having for its center the center of force, and for its principal axis the axis of rotation.

If $a$ be the radius of the sphere into which the body will have formed itself before the communication of the rotatory motion, the axes of the spheroid may be readily shown to be represented by

$$\left(\frac{m - a^2}{2m}\right)^{\frac{1}{3}} a, \quad \text{and} \ \left(\frac{2m}{m - a^2}\right)^{\frac{1}{6}} . \, a.$$

Suppose a fluid mass to be attracted towards two centers of force by forces which vary according to given functions $R$ and $R'$ of the distances $r$ and $r'$. Call $a$ the distance of the two centers of force from each other, and let $mR$ and $m'R'$ represent the actual forces upon any portion of the fluid. Take for the plane of $x\,y$, a plane passing through both centers of force. Then will the forces resolved in the directions of $x$, $y$, and $z$, be represented by

$$-\,m\,R\,\frac{x}{r}, \qquad -\,mR\,\frac{y}{r}, \qquad -\,mR\,\frac{z}{r},$$

$$+\,m'R'\,\frac{a-x}{r}, \qquad -\,m'R'\,\frac{y}{r}, \qquad -\,m'R'\,\frac{z}{r}.$$

Therefore at the surface,

$$-\,mR\left(\frac{x\,dx + y\,dy + z\,dz}{r}\right) - m'R'\left(\frac{-(a-x)\,dx + y\,dy + z\,dz}{r'}\right) = 0;$$

$$\therefore\ mR\,dr + m'R'dr' = 0.$$

The surface is clearly one of revolution. If the forces vary as the powers $n$ and $n'$ of the distances,

$$\left(\frac{m}{n+1}\right) r^{n+1} + \left(\frac{m'}{n'+1}\right) r'^{n'+1} = c.$$

If the absolute forces are the same, and vary according to the same power of the distance,

$$r^{n+1} + r'^{n+1} = c.$$

If the force be constant, $n=0$;

$$\therefore\ r + r' = c,$$

or the surface is that of a spheroid, the centers of force being its foci. If one of the forces be repulsive, one of the quantities $m$, $m'$ becomes negative, and we have

$$r - r' = c,$$

or the surface is that of an hyperboloid. If one of the centers of force be at an infinite distance, the surface resolves itself into that of a paraboloid of revolution.

Generally, if the force be constant,

$$mr + m'r' = c.$$

And this is the equation to the surface of contact of two fluids of different densities attracted simultaneously to two centers of constant force. But if the fluids be transparent, and their index of refraction equal to $\frac{m'}{m}$, the above is the equation to that surface by which rays diverging from one of the centers will be refracted to the other. It appears, then, that if two incompressible transparent fluids be attracted to two centers of constant force, the ratio of whose intensities is equal to the index of refraction, their common surface will be such as to refract light accurately from one center to the other.

106. Ex. 3. If a cylindrical vessel of homogeneous and incompressible fluid acted upon by gravity, be made to revolve about its axis, to determine the form which will be assumed by the surface of the fluid.

Taking the axis of the vessel for the axis of $z$, and its base for the plane $xy$, we have $Z =$ constant $= -g$: and

since the centrifugal force on any particle is $a^2\rho$, $\rho$ being its distance from the axis of rotation, and $a$ the angular velocity, the forces in $x$ and $y$ are respectively $a^2x$, and $a^2y$, taken positively, since they tend to increase the co-ordinates. Substituting therefore, in the general equation for the surface,

$$a^2xdx + a^2ydy - gdz = 0;$$

$$\therefore\ a^2x^2 + a^2y^2 - 2gz = c,$$

the equation to the surface of a paraboloid of revolution, having for its axis the axis of $z$.

107. If $z_{,}$ be the value of $z$ at the vertex, we have, since at this point, $x = 0$, $y = 0$,

$$-2gz_{,} = c;$$

$$\therefore\ a^2x^2 + a^2y^2 - 2g(z - z_{,}) = 0;$$

$$\text{or, } x^2 + y^2 - \frac{2g}{a^2}(z - z_{,}) = 0;$$

therefore $\dfrac{2g}{a^2}$ is the parameter of the generating parabola.

108. Let $a$ be the radius of the cylinder, and $z_{,,}$ the greatest height to which the fluid is made to ascend;

$$\therefore\ a^2 - \frac{2g}{a^2}(z_{,,} - z_{,}) = 0;$$

$$\therefore\ z_{,,} - z_{,} = \frac{a^2a^2}{2g},$$

by which quantity the lowest lies below the level of the highest portion of the fluid. Also, since the content of a paraboloid is half that of the circumscribing cylinder, the volume of the fluid displaced,

$$= \tfrac{1}{2}\pi a^2 . (z_{,,} - z_{,});$$

and therefore, the whole volume of fluid in the vessel is represented by

$$\pi a^2 z_{,} + \tfrac{1}{2}\pi a^2 (z_{,,} - z_{,}).$$

If, therefore, $h$ be the height at which it stood before motion was communicated, we have

$$\pi a^2 h = \pi a^2 z_{,} + \tfrac{1}{2}\pi a^2 (z_{,,} - z_{,});$$

$$\therefore\ z_{,,} + z_{,} = 2h.$$

Also, $$z_{,,} - z_{,} = \frac{\alpha^2 a^2}{2g};$$

$$\therefore\ z_{,,} = h + \frac{\alpha^2 a^2}{4g},$$

$$z_{,} = h - \frac{\alpha^2 a^2}{4g},$$

whence it appears that the surface of the fluid ascends above and descends below its original level by the same quantity, viz. $\dfrac{\alpha^2 a^2}{4g}$.

109. Ex. 4. A rectangular vessel containing fluid, (Fig. 12.) is made to move along a horizontal plane $TA$, by means of a weight $P$, acting over a pulley at $A$; it is required to determine the form of the surface of the fluid, and its position at any period of the motion.

The accelerating force communicated to the vessel, and in common with it to every portion of the fluid it contains, is represented by $\dfrac{Pg}{P+M}$, $M$ being the mass of the vessel and fluid. Let an equal force be supposed to be impressed upon both in an opposite direction, and from the beginning of the motion; then will the vessel remain at rest, and the (relative) position of the fluid in it at the end of any given time will be the same as though both had been in motion during that time. Suppose the surface by the action of the uniform forces now impressed upon it, in vertical and horizontal directions, to be brought into the position $KL$, and to rest in that position.

Let $QT = x$, $TN = y$, $QR = a$,

$QL = y_{,}$, $KR = y_{,,}$;

we have therefore, $X = \dfrac{Pg}{P+M}$, $Y = -g$, $z = 0$;

$$\therefore\ 0 = \frac{Pgdx}{P+M} - gdy,$$

$$c = \frac{Pgx}{P+M} - gy,$$

the equation to a straight line. Also, $c = 0 - gy_{,}$;

$$\therefore\ 0 = \left(\frac{Pgx}{P+M}\right) - g(y - y_{,});$$

$$\therefore\ y_{,,} - y_{,} = \frac{Pa}{P+M}.$$

Also, since if $b$ be the original height of the fluid, the content of the section is $a \,.\, b$, and that in its present position its content is measured by $\frac{1}{2}(y_{,} + y_{,,})\,a$, we have (the fluid being incompressible),

$$\tfrac{1}{2}(y_{,} + y_{,,})\,a = ab;$$

$$\therefore\ y_{,} + y_{,,} = 2b;$$

hence,

$$y_{,,} = b + \tfrac{1}{2}\frac{Pa}{M+P},$$

$$y_{,} = b - \tfrac{1}{2}\frac{Pa}{M+P}.$$

110. Ex. 5. Suppose a cylindrical vessel containing fluid to be made to revolve upon its axis with an uniform angular velocity ($a$), to determine the pressure upon its sides.

Let the specific gravity of the fluid be unity;

$$\therefore\ dp = a^2 x dx + a^2 y dy - g dz;$$

$$\therefore\ p = \frac{1}{2} a^2 (x^2 + y^2) - gz + c.$$

Now at that part of the surface which is immediately in contact with the coats of the vessel, let $z=z_{,}$; also let the radius of the cylinder be $a$;

$$\therefore\ 0=\frac{1}{2}\alpha^2 a^2-gz_{,}+c;$$

Also at any point in the sides of the vessel

$$p=\frac{1}{2}\alpha^2 a^2-gz+c,$$

$$\therefore\ p=g\,(z_{,}-z),$$

And this being the pressure on an unit, we have for the pressure on an elementary annulus, $2pa\pi dz$. Therefore the whole pressure on the sides

$$=2\pi ag\int(z_{,}-z)\,dz,$$

$$=\pi agz_{,}^2,$$

taken from 0 to $z_{,}$. Also $z_{,}$ has been shown (Art. 111.) to be equal to

$$\left(h+\frac{\alpha^2 a^2}{4g}\right);$$

therefore the pressure on the sides is equal to

$$\pi ag\left(h+\frac{\alpha^2 a^2}{4g}\right)^2.$$

To find the pressure on the base, we have, if $r$ be the distance of any point in the base from its center,

$$p=\frac{1}{2}\alpha^2 r^2+c.$$

Now as before, $0=\frac{1}{2}\alpha^2 a^2-gz_{,}+c$;

$$\therefore\ p=-\frac{1}{2}\alpha^2\,(a^2-r^2)+gz_{,};$$

therefore the whole pressure

$$=2\pi\int\left\{-\frac{1}{2}a^2(a^2-r^2)+gz_{,}\right\}rdr,$$

$$=\pi\left\{-\frac{1}{2}a^2a^4+\frac{1}{4}a^2a^4+gz_{,}a^2\right\} \text{ taken from } r=0 \text{ to } r=a,$$

$$=\pi a^2\left\{gz_{,}-\frac{1}{4}a^2a^2\right\}, \text{ or, substituting for } z_{,},$$

$$=\pi a^2\left\{gh+\frac{1}{4}a^2a^2-\frac{1}{4}a^2a^2\right\},$$

$$=\pi a^2hg.$$

Therefore the whole pressure on the base is equal to the whole weight of the contained fluid, as is manifest.

On the whole, therefore, the pressure on the coats of the vessel is equal to

$$\pi ag\left(h+\frac{a^2a^2}{4g}\right)^2+\pi a^2.h.g.$$

To find what must be the radius ($r$) of the vessel that the quantity of fluid ($A$) being given, the pressure sustained by the whole containing surface of the cylinder may be a maximum, we have, since $\pi r^2h=A$, and therefore $h=\dfrac{A}{\pi r^2}$,

$$\pi ag\left(\frac{A}{\pi r^2}+\frac{a^2r^2}{4g}\right)^2+Ag=\text{max}^{\text{m}}.$$

$$\therefore\left(\frac{A}{\pi r^2}+\frac{a^2r^2}{4g}\right)\left(\frac{2a^2r}{4g}-\frac{2A}{\pi r^3}\right)=0;$$

$$\therefore \frac{a^2r}{4g}-\frac{A}{\pi r^3}=0;$$

$$\therefore r^4=\frac{4Ag}{a^2};$$

$$\therefore r=\sqrt[4]{\frac{4Ag}{a^2}}.$$

# CHAP. X.

## ON THE EQUILIBRIUM OF ELASTIC FLUIDS.

111. Def. Elastic fluids are such as being compressed by any force are continually made to occupy a less space as that force is *increased*, and recover their bulk again by the *same* degrees as the force is similarly *diminished*.

112. Perfectly elastic fluids are those whose increase or diminution in bulk is exactly proportional to the diminution or increase of the force compressing them. To this class appear to belong all the aeriform fluids presented to us by nature*.

113. In every state of its density, an elastic fluid makes some effort to expand itself, and is therefore retained, when at rest, by some pressure. The density and pressure begin therefore together: and their increments are proportional; the density therefore varies as the pressure, or $p = C \cdot D$.

114. If the only force by which an elastic fluid is acted upon, be the pressure of the surface which contains it, it is apparent that this pressure disseminating itself *equally* through every portion of that fluid, the density in all such portions will be the same. And that in its state of equilibrium there will be established between the pressures on different portions of the containing surface of an elastic fluid, the same relation as obtains in the case of an inelastic fluid. Thus, if the pressure on any portions of surface $A$ and $A'$ be represented by $P$, $P'$, we have (Eq. $a$)

$$\frac{P}{A} = \frac{P'}{A'}.$$

* Elastic fluids are further distinguished into such as are permanently elastic, and such as under certain circumstances lose their elasticity and assume the form of liquids. This distinction belongs, however, rather to Chemistry than Hydrostatics.

115. Also since, by the definition of elastic fluids it appears that the bulk diminishes uniformly as the pressure increases, and conversely, also since this is true *sine limite*, it follows, that the pressure varies inversely as the volume; and calling $V$ and $V'$ the volumes of the same quantity of fluid under the pressures $P$ and $P'$,

$$PV = P'V'.$$

116. If the fluid be acted upon by forces other than the resistance of the surface which contains it, the density will be variable.

Calling $P$ the pressure on a surface $A$ in a part of the fluid whose density is $D$, and taking $P'$, $A'$, $D'$ similarly in any other portion of the fluid, we have, since the unit of pressure varies as $D$, or $= C \cdot D$,

$$P = C \cdot A \cdot D,$$
$$P' = C \cdot A' \cdot D';$$
$$\therefore \frac{P}{P'} = \frac{AD}{A'D'}.$$

117. The atmosphere which surrounds our Earth is found by experiment to be a perfectly elastic fluid.

It appears, also, that the pressure produced on any plane, taken horizontally in it, is equal to the weight of a superincumbent column (Art. 16.) Now since the weights of such columns diminish as we ascend from the Earth's surface, it is clear that the pressure on any given surface in the fluid will diminish: and the density varies as the pressure; therefore the density of the air will continually diminish as the altitude increases. Let us suppose for an instant an atmosphere of uniform density to surround the Earth, and let $h$ be its height such that the pressure at the Earth's surface may be the same as in the case of variable density which actually obtains: then at the Earth's surface we shall have, since $ghD$ is the weight of the superincumbent column, which by hypothesis is equal to the pressure,

$$p = ghD.$$

118. Now $p \propto D$ in all cases. Therefore, *generally*,

$$p = ghD.$$

$h$ is called the height of an homogeneous atmosphere, and at the mean density of the air at the Earth's surface, it is found to be 4342 fathoms.

119. If (Art. 22, Fig. 25.) $z$ represent the height $A' A''$ to which the surface $A'$ is raised above the surface $A$ by the removal of the pressure of the air above it, we have, since

$$p = ghD,$$

$$hD = zD',$$

$D'$ being the density of the fluid. If, therefore, $\sigma$ represent the ratio of the specific gravities of the air and fluid,

$$z = h \cdot \sigma.$$

120. Since $p \propto D$, in every possible state of the density of an elastic fluid, it will exert some pressure or some effort to expand itself: it can never, therefore, be held at rest unless an adequate pressure be applied to every portion of its surface. Thus if from a vessel containing an elastic fluid any portion, as for instance the superior portion, be removed, the remainder will not, as in the case of incompressible fluids, remain at rest, but will expand itself until it is again retained by some intervening surface, or some pressure otherwise supplied.

121. By equation ($\beta$) we have, in all cases of fluid equilibrium, where the accelerating force is gravity,

$$p = -\int DG\,dz.$$

Hence, therefore, where the fluid is elastic, since

$$p = c \cdot D,$$

$$cp = -\int pG\,dz;$$

$$\therefore\ c\frac{dp}{p} = -G\,dz.$$

Calling, therefore, $a$ the radius of the Earth, and $z$ the height of any portion of the atmosphere above its surface,

$$\because\ G = \frac{ga^2}{(a+z)^2};$$

$$\therefore\ c\,.\,\frac{dp}{p} = \frac{-ga^2}{(a+z)^2}\,dz;$$

$$\therefore\ c\ \text{h. l.}\ \frac{p}{\Pi} = \frac{-gaz}{(a+z)}.$$

$\Pi$ being the unit of pressure at the Earth's surface;

$$\therefore\ p = \Pi\,\epsilon^{\frac{-gaz}{c(a+z)}},$$

Since $p = cD$, and $\Pi = c\,.\,\Delta$, if $\Delta$ be the density at the Earth's surface,

$$\frac{D}{\Delta} = \frac{p}{\Pi} = \epsilon^{\frac{-gaz}{c(a+z)}},$$

$$\therefore\ D = \Delta\,.\,\epsilon^{\frac{-gaz}{c(a+z)}}.$$

Whence the density of the air at the Earth's surface being given, that at any given altitude above it is known.

122. The density of elastic fluids is subject to considerable variation from a change in their temperature. The precise nature of the agent which we call heat, or the manner of its action in the dilation of different substances, we are not acquainted with: certain it is, however, that wherever we trace its presence in a greater or less degree, we meet with a proportionate increase or diminution of bulk.

The subject is not properly one of mathematical enquiry, and we shall in the following pages confine ourselves simply to the statement of such properties of heat as are proved by experiment, and as are incidental to the proper subject of our investigation.

123. In elastic fluids it is found that, under the same pressure, equal increments of volume result, in the same quan-

tity of fluid, from equal increments of temperature, as marked by the thermometer. Hence, therefore, it follows that the relation between the temperature ($t^\circ$) and volume $V$ is expressed by the algebraical formula

$$V = a + bt^\circ \ldots\ldots\ldots\ldots\ldots (\rho)$$

in which $t^\circ$ is the variation from that temperature (as marked by the thermometer) at which the bulk of the fluid was $a$, and $b$ is the increase of that bulk for each degree of temperature.

Now $VD$ represents a quantity of matter which is given, since we are considering the variation of bulk produced by a variation in the temperature of a given quantity of fluid.

Therefore when the pressure is given,

$$D \propto \frac{1}{a + bt^\circ}.$$

Also when the temperature is given,

$$D \propto p;$$

therefore, generally,

$$D \propto \frac{p}{a + bt^\circ},$$

$$= \frac{cp}{1 + \alpha t^\circ}.$$

Where $\alpha \left(= \frac{b}{a}\right)$ is the increase of bulk in each unit of the fluid for every degree of temperature.

124. Now the temperature is found continually to diminish as we ascend from the surface of the Earth. The conclusions we have therefore deduced, with regard to the density of the air at different altitudes on the hypothesis of an equable temperature are false, and instead of assuming $D = \frac{p}{c}$, we

must take it equal to $\frac{cp}{1+at^{\circ}}$. Whence by equation ($\beta$) we have

$$p = -\int \frac{cp}{1+at^{\circ}} \cdot \frac{a^2 g}{(a+z)^2}\, dz;$$

$$\therefore \frac{dp}{p} = -\, ca^2 g \frac{dz}{(1+at^{\circ})\,(a+z)^2};$$

$$\therefore \text{h. l. } p = -ca^2 g \int \frac{dz}{(1+at^{\circ})(a+z)^2}.$$

The complete integration of this expression is impossible, since the variation of the temperature is not dependent according to any known law on that of the altitude; that is, $t^{\circ}$ cannot be expressed in terms of $z$.

125. Since, however, $a$ is exceedingly small, the variation of the quantity $1+at^{\circ}$ does not materially affect the result; and provided we take for $t^{\circ}$ the mean, $\frac{1}{2}(t^{\circ\prime}+t^{\circ\prime\prime})$, between the extreme temperatures $t^{\circ\prime}$ and $t^{\circ\prime\prime}$, we may, without sensible error, consider this temperature as common to every value of $z$*. Whence we get

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* It is found by observation, that for small altitudes, the temperature decreases very nearly in arithmetical progression as the altitude increases in that progression.

Now the quantity $\int \frac{dz}{(1+at^{\circ})(a+z)^2}$ may be considered as the sum of a series of functions of $z$ differing from one another by reason of equal increments of $z$, and each divided by a corresponding value of the quantity $(1+at^{\circ})$. Call these functions of $z$, $a, b, c \ldots l$. The corresponding values of $t^{\circ}$ are in a decreasing arithmetical progression. Let the degrees of temperature be so taken that each shall be the decrement corresponding to each equal increment $dz$ of $z$: then

$$\int \frac{dz}{(1+at^{\circ})\,(a+z)^2} = \frac{a}{1+at'} + \frac{b}{1+a\,(t'-1)} + \frac{c}{1+a\,(t'-2)} + \cdots$$
$$+ \frac{l}{1+a\,\{t'-(n-1)\}},$$

$= a$

$$\text{h. l. } p = \frac{-ca^2g}{1+\frac{1}{2}a(t'+t'')}\int\frac{dz}{(a+z)^2},$$

And calling $p'$ the pressure at the earth's surface,

$$\text{h. l. } \frac{p'}{p} = \frac{cagz}{\{1+\frac{1}{2}a(t'+t'')\}\,.\,(a+z)}\,\ldots\ldots\ldots(\sigma.)$$

Also $\dfrac{D'}{D} = \dfrac{p'}{p}$, $D'$ and $D$ being the densities corresponding to the pressures $p'$ and $p$;

$$\therefore \text{ h. l. } \frac{D'}{D} = \frac{cagz}{\{1+\frac{1}{2}a(t'+t'')\}\,(a+z)}.$$

---

$= a(1-\alpha t') + b\{1-\alpha t'+\alpha\} + c\{1-\alpha t'+2\alpha\} + \ldots + l\{1-\alpha t'+(n-1)\alpha\}$ nearly.

Now, the variation of the temperature is in a constant ratio to the variation of the altitude;

$$\therefore n-1 = t'-t^{\circ} \propto z = Cz;$$

$$\therefore \int\frac{dz}{(1+at^{\circ})\,(a+z)^2} = \int\frac{(1-at'+Caz)\,dz}{(a+z)^2}$$

$$= (1-at'-Caa)\frac{z}{a(a+z)} + Ca\text{ h.l.}\left(\frac{a+z}{a}\right)$$

$$= \frac{(1-at')z}{a(a+z)} - \frac{Caz}{a+z} - Ca\text{ h.l.}\left\{1-\frac{z}{a+z}\right\}$$

$$= \frac{(1-at')z}{a(a+z)} + \tfrac{1}{2}Ca\left(\frac{z}{a+z}\right)^2 \text{ nearly}$$

$$= \left\{1-at'+\tfrac{1}{2}\frac{Caaz}{a+z}\right\}\frac{z}{a(a+z)}$$

$$= \{1-at'+\tfrac{1}{2}Caz\}\frac{z}{a(a+z)} \text{ nearly}$$

$$= \{1-at'+\frac{a}{2}(t'-t'')\}\frac{z}{a(a+z)}$$

$$= \{1-\frac{a}{2}(t'+t'')\}\frac{z}{a(a+z)}$$

$$= \frac{z}{\{1+\frac{1}{2}(t'+t'')\}\,a(a+z)} \text{ nearly.}$$

See Appendix E.

The quantity $a$ is found by experiment to be $\frac{1}{250}$ very nearly;

$$\therefore D = D' . \epsilon^{\frac{cagz}{\left(1+\frac{t+t}{500}\right)(a+z)}} \quad \ldots\ldots\ldots\ldots\ldots(\sigma.)$$

126. The following method is by La Place.

Let $T$ be taken to represent the temperature. Then, when the pressure is given, the volume of a given quantity of air varies as $T$;

$$\therefore D \propto \frac{1}{T}, \text{ the pressure being given.}$$

$$\text{Also, } \quad D \propto p, \text{ the temperature being given;}$$

therefore, generally, $D \propto \frac{p}{T}$;

$$\therefore D = \frac{cp}{T};$$

$$\therefore \text{h. l. } p = cg \int \frac{-dz}{T},$$

considering gravity as constant.

Now, it appears from observation, that for small values of $z$, if the altitude be continually increased by the same quantity, the temperature will be uniformly diminished by the same quantity; $T$ is therefore such a function of $z$, as for small values of $z$ to decrease in arithmetical progression, as that variable increases in arithmetical progression.

And further, if $T'$ and $T''$ be the temperatures at the surface of the earth, and at the altitude $z_{,}$, $T$ is such a function of $z$, as to give for its values $O$ and $z_{,}$, the quantities $T'$ and $T''$. Hence, therefore, $T$ is such a function of $z$, as for small values of that variable, will coincide with the expression

$$T = \left\{T'^2 - z . \frac{T'^2 - T''^2}{z_{,}}\right\}^{\frac{1}{2}},$$

Since in this expression when $z=0$ $T=T'$, also when $z=z'$ $T=T''$, also for exceeding small values of $z$,

$$T = T' - \frac{1}{2} z \,.\, \frac{T'^2 - T''^2}{z_, T'},$$

which satisfies the remaining condition, namely, that $T$ should decrease, as $z$ increases in arithmetical progression, for small altitudes.

Let $$\frac{T'^2 - T''^2}{z_,} = k;$$

$$\therefore \int \frac{-dz}{T} = \int \frac{-dz}{(T'^2 - kz)^{\frac{1}{2}}}$$

$$\int \frac{-dz}{T} = \frac{2}{k}(T'^2 - kz)^{\frac{1}{2}} + c.$$

Now, when $z=0$, the integral vanishes;

$$\therefore \int \frac{-dz}{T} = \frac{2}{k}(T'^2 - kz)^{\frac{1}{2}} - \frac{2T'}{k}$$

$$= \frac{2}{k}(T - T');$$

therefore, for the whole altitude from 0 to $z_,$.

$$\int \frac{-dz}{T} = \frac{2}{k}(T'' - T')$$

$$= \frac{-2z_,}{T' + T''}.$$

Now $$T \propto a + bt^0 = c_, (1 + at^0);$$

$$\therefore \int \frac{dz}{T} = \frac{2z_,}{c_, \{2 + a(t' + t'')\}};$$

$$\therefore \text{ h. l. } \frac{p'}{p} = cg \,.\, \frac{2z_,}{2 + a(t' + t'')};$$

$$\therefore \text{h. l. } \frac{D'}{D} = \frac{cgz_{,}}{1 + \frac{1}{2}\alpha(t' + t'')};$$

which evidently coincides with the former formula, if we consider $a$ as exceeding great when compared with $z$; and therefore the gravity, as constant.

127. These formulæ are used for the mensuration of heights. The ratio $\frac{p'}{p}$ is determined by observations with the barometer, as will be explained hereafter; and thence the altitude $z_{,}$ is known by the formula

$$z_{,} = \frac{1}{cg}\left\{1 + \frac{1}{2}\alpha(t' + t'')\right\} \text{ h. l. } \left(\frac{p'}{p}\right).$$

128. Let us now apply the *general* conditions of equilibrium to the case of elastic fluids.

In considering the equation

$$dp = Dd\phi,$$

we have shewn it to be a necessary condition of equilibrium, that $D$ should be a function of the force $\phi$, that is, in the case of gravity, of the distance from the earth's center, or of the altitude $z$. It has been shewn also, that $D$ is partly a function of the temperature. It follows, therefore, that no equilibrium can exist in the atmosphere, or that it can never be wholly at rest, unless the temperature be a function of the altitude, so that at the same altitude it may be the same over every portion of the earth's surface. Now, the distribution of heat over the earth's surface, is by no means equable, the temperature about the equator being considerably higher than that taken at the same altitudes near the poles, and passing through every possible gradation in the intermediate space; influenced by an infinite variety of local and temporary causes. The atmosphere is therefore, not in a state of equilibrium. Motion is said to *prevail*, among those portions of it nearest to the earth's surface, continually towards the equator, and among the superior portions, towards the poles.

129. How far the conclusions we have drawn with regard to the density of the atmosphere when in equilibrium, are applicable to the case of continual disturbance which actually obtains, it is scarcely possible to determine otherwise than by experiment. And from this it would appear that the formula we have determined, for the variation of the density at different altitudes, very nearly obtains, when these altitudes are measured above the *same portion* of the earth's surface. The disturbance affecting equally, it would seem, the whole superincumbent column.

130. In the case of a perfectly elastic fluid of uniform temperature, the equation ($v$) becomes

$$h\,dp = p\,(X dx + Y dy + Z dz);$$

$$\therefore\ h \text{ h.l. } p = \int (X dx + Y dy + Z dz).$$

Ex. Let us take the case of a cylindrical vessel of air, made to revolve about its axis. It is required to determine the pressure on any point of the containing surface, and the density of any portion of the revolving fluid.

Let ($a$) be the radius of the base, ($c$) the altitude. Suppose the cylinder of fluid to be a column of the atmosphere,* and let $D'$ represent the density at its base, ($r$) the distance of any point ($x$, $y$, $z$,) of the fluid, from the axis of the cylinder, and $\alpha$ the angular velocity;

$$\therefore\ X = \alpha^2 x, \qquad Y = \alpha^2 y, \qquad Z = -g;$$

$$\therefore\ h \text{ h. l. } p = \int \{\alpha^2 x dx + \alpha^2 y dy - g dz\},$$

$$h \text{ h.l. } p = \frac{1}{2}\alpha^2 r^2 - gz + C_{,};$$

$$\therefore\ p = C \cdot \epsilon^{\frac{\alpha^2 r^2 - 2gz}{2h}},$$

$$D = C' \cdot \epsilon^{\frac{\alpha^2 r^2 - 2gz}{2h}}.$$

* The variation of temperature being neglected.

Now, to determine the constant $C'$, since $2\pi r dr dz D$ is the content of a solid annulus, we have for the whole fluid mass,

$$2\pi C' \iint \epsilon^{\frac{\alpha^2 r^2 - 2gz}{2h}} . r dr . dz.$$

The integral being taken from $r = 0$ to $r = a$,

$$= \frac{2\pi C' h}{\alpha^2} \int \{ \epsilon^{\frac{\alpha^2 a^2 - 2gz}{2h}} - \epsilon^{\frac{-2gz}{2h}} \} dz,$$

$$= \frac{2\pi C' h}{\alpha^2} (\epsilon^{\frac{\alpha^2 a^2}{2h}} - 1) \int \epsilon^{\frac{-gz}{h}} dz,$$

$$= \frac{2\pi C' h^2}{\alpha^2 g} (\epsilon^{\frac{\alpha^2 a^2}{2h}} - 1)(1 - \epsilon^{\frac{-gc}{h}}),$$

taken from $z = 0$ to $z = c$.

Now, the quantity of air contained, equals that which would be similarly contained in the homogeneous atmosphere,

$$= \pi a^2 h . D';$$

$$\therefore \pi a^2 h D' = \frac{2\pi C' h^2}{\alpha^2 g} (\epsilon^{\frac{\alpha^2 a^2}{2h}} - 1)(1 - \epsilon^{\frac{-gc}{h}});$$

$$\therefore C' = \frac{g \alpha^2 a^2 D'.}{2h (\epsilon^{\frac{\alpha^2 a^2}{2h}} - 1)(1 - \epsilon^{\frac{-gc}{h}})};$$

$$\therefore D = \frac{D' g \alpha^2 a^2 . \epsilon^{\frac{\alpha^2 r^2 + 2g(c-z)}{2h}}}{2h (\epsilon^{\frac{\alpha^2 a^2}{2h}} - 1)(\epsilon^{\frac{gc}{h}} - 1)};$$

also since $p = hD$;

$$\therefore p = \frac{D' g . \alpha^2 . a^2 . \epsilon^{\frac{\alpha^2 r^2 + 2g(c-z)}{2h}}}{2 (\epsilon^{\frac{\alpha^2 a^2}{2h}} - 1)(\epsilon^{\frac{gc}{h}} - 1)};$$

when $c =$ infinity, the above expressions become

$$D = \frac{g\,a^2 a^2 D' \cdot \epsilon^{\frac{a^2 r^2 - 2gz}{2h}}}{2h \cdot (\epsilon^{\frac{a^2 a^2}{2h}} - 1)},$$

and $$p = \frac{1}{2} \cdot \frac{a^2 g a^2 D' \cdot \epsilon^{\frac{a^2 r^2 - 2gz}{2h}}}{(\epsilon^{\frac{a^2 a^2}{2h}} - 1)}.$$

It is evident from these expressions, that the density will be diminished near the axis of the cylinder, and increased towards its surface; and that by increasing the angular velocity ($a$), the rarefaction of the internal and the condensation of the external air may be carried to any required extent.

ON

# HYDRODYNAMICS;

OR,

## THE MOTION OF FLUIDS.

### CHAP. I.

131. The forces lost by the different parts of a system in motion, are by D'Alembert's principle precisely such as would establish an equilibrium in it.

The forces, therefore, lost by the particles of a moving fluid are such as, being severally applied to those particles, will (under the same circumstances of form and pressure) hold the fluid at rest;* and thus we are supplied with a means of passing at once from the conditions of the *equilibrium* of a fluid to those of its *motion.*

Let $\phi$, $\phi'$, $\phi''$ represent in magnitude and direction the *effective* accelerating forces on any particle $\mu$, resolved parallel to the axes of $x$, $y$ and $z$ respectively. And let $X$, $Y$, $Z$ be the forces impressed. Then are $(X-\phi)$, $(Y-\phi')$, $(Z-\phi'')$, the forces lost. Therefore, by the general conditions of equilibrium,

$$p=\int D\{(X-\phi)dx+(Y-\phi')dy+(Z-\phi'')dz\}\ldots(A).$$

* Motion produced among the particles of a *fluid mass* by the action of any accelerating force, differs from the *free* and *unconstrained* motion of the same particles, acted on by the same force, in *this,* that the force impressed upon each particle is, in the former case, counteracted *pro tanto* by the pressure of the adjacent particles, whilst in the latter it is *wholly effective.* The difference between the effective accelerating force on any particle when its motion is unconstrained, and the effective accelerating force on the same particle, when forming part of a fluid mass, is therefore *wholly produced* by the *pressure* of that portion of the fluid in which it is found. If, therefore, this difference were applied to that portion of the fluid, it would just sustain the pressure upon it: and supposing similar forces to be similarly applied throughout, and the same pressures as before to be sustained, the whole would be in equilibrium.

132. We shall refer the more general consideration of this equation to a succeeding chapter, and confine ourselves here to that particular case in which the fluid is incompressible and homogeneous, and the motion uniform, or *the accelerating force on each particle the same, as it passes through the same point in space.*

The integral ($A$) may be taken with regard to any series of corresponding values of $x$, $y$, $z$, or with regard to any line whatever of fluid particles terminating in $\mu$.

Suppose it taken with regard to those particles which occupy the space through which $\mu$ has moved. Now, by hypothesis, the accelerating force on any one of these is precisely the same with that by which $\mu$ was impelled when at the same point in space. The integral

$$\int(\phi dx + \phi' dy + \phi'' dz),$$

taken with regard to the whole line of particles at a *given instant*, is therefore equivalent to the same integral taken with regard to the extreme particle during the different *instants* of its motion. Also $dx$, $dy$, $dz$, which in equation ($A$) have reference to *different* particles, and are taken from one point in space to another, are equivalent to $\pm dx$, $\pm dy$, $\pm dz$, when taken to represent the elementary spaces described parallel to the axes by the *same* particle. The sign $\pm$ being taken according as the motion tends to *increase* the co-ordinates or *diminish* them.

$$\therefore \ \pm \phi dx \pm \phi' dy \pm \phi'' dz = vdv,$$

$v$ being the velocity of the particle $\mu$. Therefore, generally,

$$p = D\int(Xdx + Ydy + Zdz) \mp D\int vdv.$$

Now, calling $P$ the pressure that would result if the same forces were impressed and the fluid at rest, we have

$$P = D\int(Xdx + Ydy + Zdz);$$

$$\therefore \ p = P \mp \tfrac{1}{2}{}^{*}Dv^2 + C \dots\dots\dots\dots\dots\dots (B).$$

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* See Appendix B.

Let it now be supposed that the fluid, having continually been renewed under the same circumstances, and accelerated by the same force at any the same point $(x, y, z)$ of its course, has at length attained an uniform velocity at that point. Then is $v$ the velocity of *any* particle passing through it, and therefore of a particle which has descended from the surface. Let $V$ be the velocity of this particle when at the surface of the fluid, and $p'$ the pressure on the surface;

$$\therefore\ p' = \mp \tfrac{1}{2} D V^2 + C;$$

$$\therefore\ p - p' = P \mp \tfrac{1}{2} D (v^2 - V^2) \ldots\ldots\ldots\ldots\ldots (C).$$

133. If a continuous fluid, wholly contained by the sides of a rigid vessel, be in motion, *the same quantity of fluid will pass, in the same time, through any two given sections of that vessel.* For if there enter *more* fluid by one section than escapes by the other, the intervening space will, at one time, contain *more* fluid, and if there enter *less*, *less* than at another. Both which cases are impossible, since the fluid is incompressible and the space given.

134. If the area of one section be exceedingly small as compared with that of the other, the (mean) velocity of the fluid passing through it will be exceedingly great as compared with the (mean) velocity of that passing through the other.

135. If the sections be supposed to be both perpendicular to the direction of the motion of the fluid through them, and $V$, $v$ be the mean velocities, $K$, $k$ the areas of the sections; then are $KVdt$ and $kvdt$ the quantities of fluid which pass through them in the same time $dt$;

$$\therefore\ KVdt = kvdt;$$

$$\therefore\ KV = kv.$$

136. If a stream of fluid constantly renewed and moving with an uniform velocity, be allowed to descend freely from

a given height; the descending portion of it will eventually acquire an invariable form. And, this state being attained, the same quantity of fluid will pass, in the same time, through any two horizontal sections of it. For otherwise the intervening fluid mass would alter its form, which is contrary to the supposition. Now, the velocity of the descending particles is manifestly greater in the lower sections of the stream than the higher. The lower sections are therefore *less* than the higher, and the stream contracts as it descends. And similarly it may be shown, that if the stream be thrown *upwards* it expands as it ascends.

If we conceive all the particles of each horizontal section to have descended at the same instant from rest, and to have acquired the same velocity in the descent, viz. that due to the height; and further, if we conceive the different horizontal sections of the stream to be similar planes; then, taking any vertical section of the stream for the axis of $x$, the horizontal sections will vary as $y^2$. Also, measuring $x$ from the point whence the stream falls, the velocity will vary as $\sqrt{x}$;

$$\therefore\ VK \propto y^2\sqrt{x} = \text{constant} = c;$$

$$\therefore\ y^4 = \frac{c^2}{x}.$$

Every vertical section is, therefore, *bounded* by an hyperbolic curve determined by the above equation.

If the fluid in the act of being let fall, be projected with a velocity due to the height $a$, we shall have

$$y^4 = \frac{c^2}{a \pm x}.$$

The sign $\pm$ being taken according as the fluid is projected upwards or downwards.

Thus it appears, that the stream thrown up by a fountain, is nearly that formed by the revolution of a hyperbolic curve of the fourth order about an asymptote.

# CHAP. II.

## ON THE MOTION OF FLUIDS THROUGH SMALL ORIFICES.

137. Let the medium into which the efflux takes place be supposed the same with that, in contact with the surface of the fluid, as where the vessel is open and immersed in the atmosphere. The pressure at the orifice and surface are in this case the same.

We have, therefore, (Art. 132.) if $v$ be the velocity at the orifice, and $P$ an unit of the pressure, it would sustain, if closed,

$$0 = P - \tfrac{1}{2} D (v^2 - V^2);$$

$$\therefore\ v^2 = V^2 + 2\int (X dx + Y dy + Z dz).$$

138. Suppose the fluid to be acted upon by the constant force of gravity, and to flow through an orifice any where situated in the containing vessel. Now, if it be kept continually at the same height, the condition of uniform acceleration, we have supposed, will manifestly obtain; and we shall have $v^2 = V^2 + 2gz$, $v$ being the velocity of any particle of the issuing fluid.

If the fluid be not supplied at the same rate in which it escapes, the position of the surface will no longer be stationary, and the hypothesis of uniform acceleration will not obtain. Since, however,

$$\sqrt{V^2 + 2gz}$$

is the velocity which would be acquired at the orifice, if the surface remained stationary during a certain time in any one of its positions; it is clear, that the actual velocity will approximate continually to this, as the motion of the surface takes place more slowly, or as it remains longer

in each of the positions it assumes. That is, according as the efflux approaches more nearly to the influx, or, if there be no influx, according as the aperture is less, when compared with the surface of the fluid. Now this approximation may be carried on *sine limite*. Also, when the area of the aperture vanishes as compared with the surface of the fluid, $V$ vanishes as compared with $v$. Therefore, on the whole,

$$v = \sqrt{2gz}$$

is a *limit* never actually attained by the velocity in any position of the surface, but continually approximated to, as the area of the aperture is less when compared with it.

It will be observed, that the velocity $v$ is that which is said to be *due* to the height ($z$) of the fluid above the orifice, or which would be acquired by a body falling freely through that height.

139. If the aperture be so contrived as to direct the stream of issuing fluid obliquely upwards, and the surface be kept continually at the same height, each particle of the issuing fluid (supposed to be projected freely in space with the velocity due to the height) will be made to describe the same parabolic trajectory, and the whole jet will assume the form of a parabola. The range on a horizontal plane passing through the aperture being represented by $2z \sin \alpha$, the height by $z \sin^2 \alpha$, and the time of flight by $2\sqrt{\frac{2z}{g}} \cdot \sin \alpha$. (See *Whewell*, 238.)

140. The point where a jet will meet a horizontal plane situated *beneath* it, may be determined by substituting the distance of the plane beneath the jet, for $y$ in the general equation to the trajectory, (*Whewell*, 240.), and solving with regard to $x$.

Let it be required to determine where an aperture must be made in the side of a prismatic vessel of fluid, that the jet may strike a given point in the plane on which it stands.

Take $a$ to represent the height of the surface of the fluid above the plane, and $y$ the height of the orifice; then is $a-y$ the height due to the velocity at the orifice: and calling $b$ the distance of the given point from the base of the vessel,

$$-y = b \tan \alpha - \frac{b^2}{4(a-y)\cos^2 \alpha};$$

whence we obtain

$$y = \tfrac{1}{2}\left\{(a - b \tan \alpha) \pm \sqrt{a^2 + 2ab \tan \alpha - b^2}\right\}.$$

If this expression be possible, there are two positions of the orifice, for which, the point where the jet strikes the horizontal plane will be the same, (see Fig. 36.) the angle of elevation being the same in both cases.

141. If by reason of the descent of the surface or other causes, which will be hereafter explained, the particles which at any time form a part of the jet, have *not* all been projected from the orifice with the same velocity; different parabolas, $APC$, $AQB$ (Fig. 35.) will be described, and the stream will cease to be continuous; the dispersion being greatest near the extremity of the range.

*To find the time in which a vessel will empty itself through an exceeding small aperture in its base.*

142. If $K$ represent at any time the area of the surface of the fluid, $k$ a section of the aperture, $v$ the velocity of effluence, and $dz$ the descent of the surface during the increment of time $dt$; then is $Kdz$ the quantity by which the fluid in the vessel is diminished in the time $dt$, and $kvdt$ is the quantity which flows through the aperture;

$$\therefore\ kvdt = -Kdz.$$

The sign $-$ indicating, that the height $z$ of the surface is diminished by $dz$.

Now, $\quad v = \sqrt{2gz}; \qquad \therefore\ t = \dfrac{1}{k\sqrt{2g}} \int \dfrac{-K dz}{\sqrt{z}}.$

Ex. 1. Suppose the vessel to be prismatic, so that the horizontal section $K$ may be the same throughout;

$$\therefore\ t = \frac{K}{k\sqrt{2g}} \int \frac{-dz}{\sqrt{z}} = \frac{2K}{k\sqrt{2g}} (\sqrt{a} - \sqrt{z}) \ldots\ldots (1),$$

where $a$ is the value of $z$ at the beginning of the motion. For the *whole* time of efflux we have

$$t = \frac{2K\sqrt{a}}{k\sqrt{2g}}.$$

Now, if the surface had been kept continually at the same height ($a$), and the *same quantity* of fluid as in the former case had flowed out in the time $t'$, since the velocity of efflux would have been uniformly equal to $\sqrt{2ga}$, we should have had

$$t' k\sqrt{2ga} = Ka;$$

$$\therefore\ t' = \frac{Ka}{k\sqrt{2ga}} = \tfrac{1}{2} t.$$

143. From equation (1) we obtain

$$\sqrt{z} = \sqrt{a} - \frac{k\sqrt{2g}}{2K} t;$$

$$\therefore\ a - z = \frac{k\sqrt{2ga}}{K} t - \frac{k^2 g}{2K^2} t^2.$$

Now, the right-hand member of this equation is the expression for the space which would be described by a body projected with the velocity $\dfrac{k\sqrt{2ga}}{K}$, and retarded by the *constant* force $\dfrac{k^2 g}{K^2}$. Also, $\sqrt{2ga}$ is the velocity at the aperture,

and therefore $\frac{k\sqrt{2ga}}{K}$ that of the surface at the beginning of the time $t$, and $a-z$ is the space described by the surface. The motion of the surface is, therefore, retarded by the constant force $\frac{k^2 g}{K^2}$.

Ex. 2. To find the time in which an ellipsoid will empty itself through a given exceedingly small aperture in its vertex, when placed with its greatest axis in a vertical position.

Let $a$, $b$, $c$ be the semi-axes of the ellipsoid, and $x$, $y$ the semi-axes of the elliptical *surface* of the fluid, at any given period of the efflux. Also, let $z$ be the distance of the surface from the center of the ellipsoid;

$$\therefore\ x = \frac{b}{a}(a^2 - z^2)^{\frac{1}{2}}, \qquad y = \frac{c}{a}(a^2 - z^2)^{\frac{1}{2}};$$

$$\therefore\ K = \frac{4}{3}\,\frac{\pi b c}{a}(a^2 - z^2);$$

$$\therefore\ k\sqrt{2g(a-z)}\,.\,dt = \frac{4}{3}\,\frac{\pi b c}{a}(a^2 - z^2)\,dz;$$

$$\therefore\ t = \frac{4\pi b c}{3ka\sqrt{2g}}\int (a-z)^{\frac{1}{2}}(a+z)\,dz$$

$$= \frac{4\pi b c}{3ka\sqrt{2g}}\int \{2a(a-z)^{\frac{1}{2}} - (a-z)^{\frac{3}{2}}\}\,dz$$

$$= \frac{4\pi b c}{3ka\sqrt{2g}}\left\{\frac{4}{3}a(2a)^{\frac{3}{2}} - \frac{2}{5}(2a)^{\frac{5}{2}}\right\} = \frac{64\pi b c a^{\frac{3}{2}}}{45k\sqrt{g}}.$$

If the aperture $k$ be supposed to be formed by a section of the vessel at an exceeding small distance $\kappa$ from its vertex,

$$k = \frac{4}{3}\,\frac{\pi b c}{a}\{a^2 - (a-\kappa)^2\} = \frac{8}{3}\pi b c \kappa \text{ very nearly};$$

$$\therefore\ t = \frac{8a^{\frac{3}{2}}}{15\kappa\sqrt{g}}.$$

The time is, therefore, on this hypothesis, independent of the magnitude of the axes $b$ and $c$, and is the same in the ellipsoid, spheroid, and sphere.

Ex. 3. To find the time of emptying a vessel, formed by the revolution of a cycloid about its axis, through a small aperture in its vertex.

$$t = -\frac{\pi}{k\sqrt{2g}}\int\frac{y^2\,dx}{\sqrt{x}};$$

$$\therefore\ \frac{-t\,.\,k\sqrt{2g}}{\pi} = \int y^2 x^{-\frac{1}{2}}dx = 2y^2x^{\frac{1}{2}} - 4\int yx^{\frac{1}{2}}dy,$$

$$\int yx^{\frac{1}{2}}dy = \int y\sqrt{2a-x}\,dx = -\frac{2}{3}\,.\,y(2a-x)^{\frac{3}{2}} + \frac{2}{3}\int(2a-x)^{\frac{3}{2}}dy,$$

$$\int(2a-x)^{\frac{3}{2}}dy = \int\frac{(2a-x)^2}{x^{\frac{1}{2}}}dx = 8a^2x^{\frac{1}{2}} - \frac{8}{3}ax^{\frac{3}{2}} + \frac{2}{5}x^{\frac{5}{2}} + C;$$

$$\therefore\ t = \frac{-\pi}{k\sqrt{2g}}\left\{2y^2x^{\frac{1}{2}} + \frac{8}{3}y(2a-x)^{\frac{3}{2}} - \frac{8}{3}\left(8a^2x^{\frac{1}{2}} - \frac{8}{3}ax^{\frac{3}{2}} + \frac{2}{5}x^{\frac{5}{2}}\right)\right\} + C.$$

And taking this integral from

$$x = 2a,\ y = \pi a \quad \text{to} \quad x = 0,\ \ y = 0,$$

we obtain for the whole efflux

$$t = \frac{\pi a^{\frac{5}{2}}}{k\sqrt{g}}\left\{2\pi^2 - \frac{8^3}{45}\right\}.$$

Ex. 4. A vertical cylinder of fluid revolves uniformly about its axis; to find the time of efflux through an exceeding small aperture in its side.

Let $\alpha$ represent the angular velocity and $a$ the radius of the cylinder. Also, let the height of the lowest point of the surface of the fluid above the aperture, be represented by $z$.

Then (Art. 106.)

$$\int(Xdx + Ydy + Zdz) = \tfrac{1}{2}a^2\alpha^2 + gz;$$

therefore,

$$v = \sqrt{a^2\alpha^2 + 2gz}.$$

Now it has been shown, (Art. 108.) that if $A$ be the quantity of fluid at any time contained in the vessel,

$$z = \frac{A}{\pi a^2} - \frac{a^2\alpha^2}{4g};$$

$$\therefore\ \pi a^2 dz = dA; \qquad \therefore\ kvdt = -\pi a^2 dz;$$

$$\therefore\ t = \frac{\pi a^2}{k}\int\frac{-dz}{\sqrt{a^2\alpha^2 + 2gz}} = \frac{\pi a^2}{kg}\left\{\sqrt{a^2\alpha^2 + 2gz} - a\alpha\right\}.$$

Taking the integral from $z = c$ to $z = 0$.

Ex. 5. A vessel containing fluid which flows out through a small aperture in its base, is drawn vertically upwards by means of a weight (Fig. 37.) acting over pullies. To determine the motion.

Let $K$ represent the surface of the fluid and $k$ the aperture, $z$ the height of the surface above the aperture, and $f$ the accelerating force on the vessel and fluid upwards. Also, let $C$ be the sum, and $C'$ the difference, of the mass of the vessel (without the fluid), and the mass $P$ of the weight. Then is the accelerating force on the system represented by

$$g \cdot \frac{C' - \int K dz}{C + \int K dz} = f.$$

Suppose this force to have been communicated in an opposite direction to the vessel and fluid in the beginning of the motion. The motion of the fluid, with respect to the vessel, will be the same on this hypothesis as in the case which actually obtains, and the vessel will be at rest. Hence, since $g + f$ is the whole accelerating force impressed downwards, on the above hypothesis, and that the aperture is exceeding small, we have

$$v^2 = 2\int(g+f)\,dz = 2gz + 2g\int\frac{C' - \int K\,dz}{C + \int K\,dz}\,dz$$

$$= 4Pg\int\frac{dz}{C + \int K\,dz}.$$

Suppose the figure a paraboloid of revolution. Hence

$$\int K\,dz = \pi c z;$$

$$\therefore\ v^2 = 4Pg\int\frac{dz}{C + \pi c z^2} = \frac{4Pg}{C}\sqrt{\frac{C}{\pi c}}\,.\,\tan^{-1} z\sqrt{\frac{\pi c}{C}}$$

$$= \frac{4Pg}{C}\left\{z - \frac{1}{3}z^3\left(\frac{\pi c}{C}\right) + \frac{1}{5}z^5\left(\frac{\pi c}{C}\right)^2 - \right\}.$$

If $\pi c z^3$, &c. be exceeding small, as compared with $C$, $v = 2\sqrt{\dfrac{Pgz}{C}}$, or it equals the velocity with which the fluid would escape, if the vessel were at rest, multiplied by $\sqrt{\dfrac{2P}{C}}$. The time in the former case will, therefore, be found by dividing that in the latter by $\sqrt{\dfrac{2P}{C}}$.

144. If we conceive a vessel, from which fluid escapes through a small aperture, to be continually supplied by a stream, moving at any given time with a velocity $v'$, and furnishing in an unit of time a quantity of fluid represented by $k'v'$; the fluid contained in the vessel is on the whole *increased* in the time $dt$ by $(k'v' - kv)\,dt$;

$$\therefore\ (k'v' - kv)\,dt = K\,dz;$$

$$\therefore\ t = \int\frac{K\,dz}{k'v' - k\sqrt{2gz}}.$$

Ex. 1. If the vessel be prismatic and the influx constant, $K$ and $k'v'$ are given;

$$\therefore\ t = \frac{K}{k\sqrt{2g}}\int\frac{dz}{\dfrac{k'v'}{k\sqrt{2g}} - \sqrt{z}}.$$

Let $$\frac{k'v'}{k\sqrt{2g}} = \sqrt{a};$$

$$\therefore\ t = \frac{2K}{k\sqrt{2g}}\int\frac{z^{\frac{1}{2}}dz^{\frac{1}{2}}}{\sqrt{a}-\sqrt{z}} = \frac{2K}{k\sqrt{2g}}\int\left\{-dz^{\frac{1}{2}} + \frac{\sqrt{a}\,dz^{\frac{1}{2}}}{a^{\frac{1}{2}}-z^{\frac{1}{2}}}\right\}$$

$$= \frac{2K}{k\sqrt{2g}}\left\{\sqrt{z_{,}} - \sqrt{z} + \sqrt{a}\ \text{h. l.}\ \frac{\sqrt{a}-\sqrt{z_{,}}}{\sqrt{a}-\sqrt{z}}\right\},$$

taking the integral from $z_{,}$ to $z$.

We may conceive a certain time when the efflux shall equal the influx, and the surface of the fluid become stationary. Now, when this takes place,

$$k\sqrt{2gz} = k'v'; \qquad \therefore\ \sqrt{z} = \frac{k'v'}{k\sqrt{2g}} = \sqrt{a}.$$

But when $z = a$, $t = \infty^{\text{ty}}$. The surface of the fluid will not, therefore, in the above instance, become stationary in any *finite* time.

Ex. 2. Let us suppose the influx to take place from a small aperture in a cylinder containing a given quantity of fluid.

If $k'$ be the aperture in the cylinder, and $v'$ the velocity, the influx

$$= k'v' = k'\sqrt{2gz} = k'\sqrt{2ga} - \frac{gk'^2}{K}\,t, \text{ (Art. 142. Ex. 1.).}$$

Also, if $v$ and $k$ represent the velocity and aperture in the vessel,

$$(v'k' - vk)\,dt = K\,dz;$$

$$\therefore\ \left\{k'\sqrt{2ga} - \frac{gk'^2}{K'}\,t - k\sqrt{2gz}\right\}dt = K\,dz.$$

By the solution of this differential equation $z$ is known in terms of $t$.

If the surface of the vessel be formed by the revolution of the hyperbolic curve, whose equation is $y^4 z = c$, about its asymptote,

$$K = \pi \sqrt{\frac{c}{z}};$$

$$\therefore\ d\sqrt{z} + \frac{k}{2\pi}\sqrt{\frac{2g}{c}}.\sqrt{z}.dt = \left(\frac{k'}{2\pi}\sqrt{\frac{2ga}{c}} - \frac{gk'^2}{2\pi K'\sqrt{c}}t\right)dt,$$

a linear equation, which may readily be integrated.

*On the Motion of Fluids through small Apertures in a* System *of communicating Vessels.*

145. Suppose three vessels to be placed one above another, and let them communicate by small horizontal apertures in their bases. Let the height of the fluid in the highest vessel be $z$, and the distance between its base and that of the second vessel $z_1$; and similarly, let $z_2$ be the distance between the base of the second vessel and that of the third.

Let $v_1$ be the velocity of efflux from the first into the second vessel, $v_2$ that from the second into the third, and $v$ the final velocity of efflux. Also, let $p_1$, $p_2$, $p$ be the pressures at the orifices of the vessels respectively, $p$ being the unit of atmospheric pressure; and let the base of each vessel be supposed to be immersed in the fluid contained by that beneath it;

$$\therefore\ p_1 - p = Dgz - D\tfrac{1}{2}v_1^2, \qquad p_2 - p_1 = Dgz_1 - D\tfrac{1}{2}v_2^2,$$
$$p - p_2 = Dgz_2 - D\tfrac{1}{2}v^2;$$

therefore, by addition,

$$gz + gz_1 + gz_2 - \tfrac{1}{2}(v_1^2 + v_2^2 + v^2) = 0.$$

Now, if we suppose fluid to be uniformly supplied to the upper vessel, and the surfaces of the fluid in the other

vessels to have become stationary, we shall have, calling $k_1$, $k_2$, $k$ the areas of the apertures,

$$kv = k_1 v_1 = k_2 v_2;$$

$$\therefore\ gz + gz_1 + gz_2 - \tfrac{1}{2}k^2v^2\left(\frac{1}{k^2} + \frac{1}{k_1^2} + \frac{1}{k_2^2}\right) = 0;$$

$$\therefore\ v = \frac{\sqrt{2g}}{k}\sqrt{\frac{z + z_1 + z_2}{\dfrac{1}{k^2} + \dfrac{1}{k_1^2} + \dfrac{1}{k_2^2}}}.$$

And similarly if there be any number of vessels, and $h$ be the height of the base of the highest above that of the lowest, and $z$ the height at which the fluid stands in the highest, we shall find

$$v = \frac{\sqrt{2g}}{k}\sqrt{\frac{z+h}{\Sigma\left(\frac{1}{k}\right)^2}}.$$

If the influx be given, $vk$ is given $(=c)$;

$$\therefore\ z = \frac{c^2}{2g}\Sigma\left(\frac{1}{k}\right)^2 - h.$$

146. Let two vessels (Fig. 35.) communicate by means of a common aperture $P$, so that the fluid may ascend in $A$, whilst it descends in $B$.

Now, the motion in $A$ tends to increase the height $z$ of the surface above the orifice, whilst the motion in $B$ tends to diminish it: therefore

$$p' + (gz_1 + \tfrac{1}{2}v^2)D$$

represents the pressure at the orifice in the former vessel; and

$$p' + (gz_2 - \tfrac{1}{2}v^2)D$$

that in the latter, $p'$ being the unit of atmospheric pressure,

and $z_1$, $z_2$ the heights of the surfaces above the common orifice;

$$\therefore\ p' + (gz_1 + \tfrac{1}{2}v^2)\,D = p' + (gz_2 - \tfrac{1}{2}v^2)D;$$

$$\therefore\ v^2 = g\,(z_2 - z_1)^*.$$

147. To find the time in which a given quantity $A$ of fluid, poured into one of the vessels and flowing through the aperture, will attain the same level in both.

Calling $K_1$, $K_2$ the areas of the surfaces of the fluid at any given time, we have

$$\int K_1 dz_1 + \int K_2 dz_2 = A \ldots\ldots\ldots\ldots (1),$$

$$\text{also,}\qquad kvdt = K_1 dz_1;$$

$$\therefore\ t = \frac{1}{k\sqrt{g}}\int \frac{K_1 dz_1}{\sqrt{z_2 - z_1}} \ldots\ldots\ldots\ldots (2).$$

Also $K_1$ and $K_2$ are given in terms of $z_1$, $z_2$, since the form of the vessel is given; hence $z_2$ may be eliminated from these equations, and the value of $t$ found in terms of $z_1$. Now, the particular value of $z_1$ for which it equals $z_2$, is given by equation (1); therefore, &c.

Ex. 1. Suppose the vessels to be both prismatic;

$$\therefore\ K_1 z_1 + K_2 z_2 = A, \text{ and } t = K_1 \frac{\sqrt{K_2}}{k\sqrt{g}} \int \frac{dz_1}{\sqrt{A - (K_1 + K_2)\,z}};$$

which integral, taken from $z_1 = 0$ to $z_1 = \dfrac{A}{K_1 + K_2}$, gives

$$t = \frac{K_1\sqrt{K_2 A}}{k\,(K_1 + K_2)\sqrt{g}}.$$

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* It is clear that this demonstration will apply to the case in which the vessels are separate, the apertures being joined by a horizontal tube.

Ex. 2. Let us suppose the prism into which the influx takes place, to be closed at the top, and let $a$ be its height. Then, if $p''$ represent the unit of pressure on the surface of the fluid in this vessel when its height is $z_1$, and $p'$ the unit of atmospheric pressure, we have

$$p''(a-z_1)=p'a; \qquad \therefore\ p''=\frac{p'a}{a-z_1}.$$

Now, $p''+(gz_1+\frac{1}{2}v^2)D=p'+(gz_2-\frac{1}{2}v^2)D$;

$$\therefore\ v^2=g(z_2-z_1)-\frac{p'z_1}{(a-z_1)D}.$$

Or, if the height of a column of the fluid whose weight is the same with the atmospheric pressure on an area equal to its base, be represented by $h$; since

$$p'=ghD; \qquad \therefore\ v^2=g(z_2-z_1)-\frac{ghz_1}{(a-z_1)}.$$

Now, $\quad K_1z_1+K_2z_2=A$;

therefore, eliminating $z_2$,

$$v^2=g\,\frac{(K_1+K_2)z_1^2-\{K_1a+K_2(a+h)+A\}z_1+Aa}{K_2(a-z_1)}.$$

If we make $v=0$, we shall obtain two positions of the surface in which the velocity of influx will vanish.

148. If there be any number of vessels communicating as above, and having their common orifices in the same horizontal plane; calling $K_n$, $z_n$, $k_n$, $v_n$ respectively, the area of the fluid in the $n^{\text{th}}$ vessel, its height above the orifice, the area of the orifice, and the velocity of the fluid passing through it; also taking similar symbols with regard to the other vessels; since

$$k_nv_ndt-k_{n-1}v_{n-1}dt$$

is the excess of the fluid which *enters* the $n^{\text{th}}$ vessel above that which escapes from it in the time $dt$, we shall have

$$(k_n v_n - k_{n-1} v_{n-1})\, dt = -K_n dz_n \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Also, $$v_n^2 = g(z_{n-1} - z_n) \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

And calling $A$ the whole volume of the contained fluid,

$$\Sigma \int K_n dz_n = A \ldots\ldots\ldots\ldots\ldots\ldots (3).$$

Now there are $n-1$ of each of the equations (1) and (2). We have, therefore, on the whole, $2n-1$ equations; by means of which any one of the $2n$ quantities, $t$, $v_n$, $z_2$, &c. may be determined in terms of any other of them.

149. In the case in which the system consists of two vessels, let us suppose *a stream of fluid to flow continually into one of them*, and let the quantity supplied in an unit of time be, at any period of the motion, $av$;

$$\therefore\ K_1 dz_1 + K_2 dz_2 = av\,dt;$$

also $$k\sqrt{g(z_2 - z_1)}\ .\ dt = K_1 dz_1.$$

Now, since the circumstances of the influx are given, $v$ is given in terms of $t$. We may therefore eliminate one of the quantities, $z_1$, $z_2$, $t$, between the above equations, and establish a relation between the remaining two.

Eliminating $dt$,

$$K_1 dz_1 + K_2 dz_2 = \frac{av K_1 dz_1}{k\sqrt{g(z_2 - z_1)}}.$$

If the vessels be prismatic and the influx uniform, this equation may be rendered integrable by assuming

$$z_2 - z_1 = u^2.$$

If the influx be such as to keep the fluid constantly at the same height in the vessel into which it flows, $z_2$ is constant; and the equation

$$t = \frac{1}{k\sqrt{g}} \int \frac{K_1 dz_1}{\sqrt{z_2 - z_1}},$$

may be integrated immediately. If the vessel be prismatic,

$$t = \frac{2K_1}{k\sqrt{g}}\{(z_2 - z_3)^{\frac{1}{2}} - (z_2 - z_1)^{\frac{1}{2}}\},$$

$z_3$ being the value of $z_1$ at the beginning of the motion.

150. Let us now take the case in which the fluid *escapes* through a small aperture in the last vessel. First, let there be two vessels. Then, since the whole quantity of fluid which has been poured into the first vessel is equal to that which is at present contained, together with that which has escaped,

$$\int K_1 dz_1 + \int K_2 dz_2 + k\int v_2 dt = A.$$

Also, as before,

$$v_1^2 = g(z_1 - z_2), \quad (k_2 v_2 - k_1 v_1) dt = - K_2 dz_2, \quad v_2^2 = 2gz_2.$$

Whence, by eliminating and reducing, we obtain

$$\frac{K_1 dz_1}{k_1\sqrt{z_1 - z_2}} + \frac{K_2 dz_2}{k_1\sqrt{z_1 - z_2} - k_2\sqrt{2z_2}} = 0.$$

If the vessels be prismatic, the variables will be separated, and the equation rendered rational by assuming

$$z_1 = z_2(u^2 + 1).$$

151. If the surface of the fluid in the first vessel be kept constantly at the same height,

$$z_1 = \text{constant} = a;$$

$$\therefore\ t = \int \frac{-K_2 dz_2}{k_2 v_2 - k_1 v_1} = \int \frac{K_2 dz_2}{k_1\sqrt{g(a - z_2)} - k_2\sqrt{2gz_2}}.$$

If there be any number ($n$) of vessels, it may be shewn as before, that

$$\Sigma\int K dz + k\int v_n dt = A, \quad \text{and} \quad v_n = \sqrt{2gz_n}.$$

These, together with the equations (1) and (2) of Art. 148. make up $2n$ equations, determining any one of the $2n+1$ quantities, $t$, $z_n$, $v_n$, &c. in terms of any other of them.

# CHAP. III.

## THE MOTION OF FLUIDS ON THE HYPOTHESIS OF PARALLEL SECTIONS.

### *Uniform Motion through an Aperture of finite Dimensions.*

152. According to the hypothesis of parallel sections, the velocities of all the particles in any horizontal section of a descending fluid, are at any given instant the same; so that in their descent the *same* particles are *continually* found in planes which are horizontal, and therefore *parallel* to one another.

We have shown, that on this hypothesis, if $V$ and $v$ be the velocities of the particles in any two descending sections, $K$ and $k$, $KV=kv$.

153. Let us take the case of a vessel having an horizontal aperture in its base, to which the fluid is continually supplied at the same rate, so that its surface may eventually become stationary and its motion uniform. Now, when this state of uniform motion is attained, we have (Art. 132.)

$$p-p'=P-\tfrac{1}{2}D(v^2-V^2);$$

or, since $KV=kv$,*

$$p-p'=P-\tfrac{1}{2}Dv^2\left(1-\frac{k^2}{K^2}\right).$$

At the orifice $p=p'$;

$$\therefore\ v^2=\frac{2P}{D\left(1-\frac{k^2}{K^2}\right)}.$$

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* In this theorem, it is only necessary to suppose all the particles of the *higher* and *lower* sections to move with the same velocities.

If the fluid descend by the force of gravity, $P = Dgz$, and

$$v^2 = \frac{2gz}{1 - \frac{k^2}{K^2}}; \qquad \therefore\ z = \frac{v^2}{2g}\left(1 - \frac{k^2}{K^2}\right).$$

Ex. 1. Fluid is supplied with a given uniform velocity to a vessel in the form of a truncated paraboloid: it is required to find in what position its surface will become stationary.

Let $I$ represent the influx in an unit of time, $a$ the distance of the aperture from the vertex of the paraboloid;

$$\therefore\ \frac{k}{K} = \frac{a}{a+z}.$$

Also, the motion being uniform, the efflux is equal to the influx, or $vk = I$. Now, $k = \pi ca$, $c$ being the parameter of the paraboloid;

$$\therefore\ v = \frac{I}{\pi ca};$$

$$\therefore\ \frac{I^2}{\pi^2 c^2 a^2} = \frac{2gz}{1 - \left(\frac{a}{a+z}\right)^2} = \frac{2g(a+z)^2}{2a+z}.$$

Let $\quad \dfrac{I^2}{2\pi^2 c^2 a^2 g} = 2\gamma; \qquad \therefore\ \dfrac{(a+z)^2}{2a+z} = 2\gamma;$

whence $\qquad z = (\gamma - a) \pm \sqrt{\gamma^2 + 2a\gamma}.$

Ex. 2. Where must a semi-ellipsoidal vessel be truncated, that being kept constantly filled, the efflux in a given time may be the greatest possible?

Let $z$ be the distance from the center at which the required section must be made, and $a$, $b$, $c$ the semi-axes of the ellipsoid. Now, (Art. 62. Ex. 4.)

$$k = \frac{\pi ab}{c^2}(c^2 - z^2), \quad \text{and} \quad K = \pi ab; \qquad \therefore\ \frac{k}{K} = \frac{c^2 - z^2}{c^2};$$

$$\therefore\ v^2 = \frac{2gz}{1-\left(\frac{c^2-z^2}{c^2}\right)^2} = \frac{2gc^4}{2cz-z^3};$$

$$\therefore\ v^2k^2 = \frac{2g\pi^2a^2b^2(c^2-z^2)^2}{2c^2z-z^3} = \text{max.};\quad \therefore\ \frac{(c^2-z^2)^2}{2c^2z-z^3} = \text{max.};$$

whence we obtain, by the usual methods,

$$z^4 - 3c^2z^2 - 2c^4 = 0,\quad \text{or}\quad z = \sqrt{\frac{3 \pm \sqrt{17}}{2}}\,.\,c.$$

If it be required to determine where the section must be made, that the *velocity* of efflux may be the greatest possible, we shall have $\frac{2gc^4}{2c^2z-z^3} = \text{max.}$; whence $z = c\sqrt{\frac{2}{3}}$.

*Uniform Motion through a vertical Aperture of finite Dimensions.*

154. We have hitherto supposed the aperture to be *horizontal*. Let us now conceive fluid to escape by a *vertical* aperture of finite dimensions.

Let $y$ be any horizontal ordinate of the aperture, $z$ its depth, and $v$ the corresponding velocity of the effluent fluid. Then is $vydz$ the efflux through an element of the aperture, and $\int vydz$ is the *whole* efflux, referred to an unit of time. Therefore, each particle of the *surface* of the fluid being supposed to descend with the same velocity, $KV = \int vydz$.

$$\text{Now,}\qquad v^2 = V^2 + 2gz;$$

$$\therefore \int y\,(V^2+2gz)^{\frac{1}{2}}\,dz = KV.$$

Ex. 1. Let the aperture be rectangular;

$$\therefore\ y = \text{const.} = c;\quad \int y\,(V^2+2gz)^{\frac{1}{2}}\,dz = c\int (V^2+2gz)^{\frac{1}{2}}\,dz$$

$$= \frac{c}{3g}\left\{\left\{V^2+2g\,(z_1+a)\right\}^{\frac{3}{2}} - (V^2+2gz_1)^{\frac{3}{2}}\right\};$$

taking the integral from $z_1$ to $z_1 + a$; where $z_1$ is the depth of the top of the aperture, and $a$ its length;

$$\therefore \frac{c}{3g}\left\{\left\{V^2 + 2g(z_1 + a)\right\}^{\frac{3}{2}} - (V^2 + 2gz_1)^{\frac{3}{2}}\right\} = KV.$$

From this equation $V$ may be determined, the position of the surface being given; or the position of the surface may be found, the influx $KV$ being given.

Ex. 2. Suppose $y$ to vary inversely as $v$, so that $y = \frac{a}{v}$;

$$\therefore \alpha \int dz = K\sqrt{v^2 - 2gz};$$

and taking the integral throughout the length $a$ of the aperture,

$$\alpha a = K\sqrt{v^2 - 2gz};$$

$$\therefore \frac{\alpha^2 a^2}{K^2} = v^2 - 2gz = \frac{a^2}{y^2} - 2gz;$$

$$\therefore \frac{a^2}{y^2} = \frac{\alpha^2 a^2}{K^2} + 2gz; \qquad y^2 = \frac{K^2 a^2}{2K^2 gz + \alpha^2 a^2}.$$

*On uniform Motion in communicating Vessels.*

155. Suppose the vessels $A$ and $B$ to communicate at their bases; and let a fluid acted upon by gravity be supplied so as to remain constantly at the same height in $A$, whilst it flows over the sides of $B$. Now, if $z$ be measured from the surface of $A$, it is clear that the motion in the vessel $A$ tends to increase that quantity, and that in $B$ to diminish it: we have, therefore, calling $V$ and $v$ the velocities of the surfaces in the two vessels, $p'$ the unit of atmospheric pressure on either surface, and $z$ the difference of their altitudes; in the vessel $A$, $p' = -\frac{1}{2}V^2 + C$, and in the vessel $B$, $p' = -gz + \frac{1}{2}v^2 + C$. (Art. 132.)

$$\therefore gz - \tfrac{1}{2}(v^2 + V^2) = 0.$$

Now, if we suppose the particles in the two surfaces to move all of them with the same velocity,

$$VK = vk; \qquad \therefore\ v^2 = \frac{2gz}{1 + \frac{k^2}{K^2}};$$

$$\text{therefore, the efflux} = vk = \sqrt{\frac{2gz}{\left(\frac{1}{k^2} + \frac{1}{K^2}\right)}}.$$

If the surface $K$ be infinite as compared with $k$, $v^2 = 2gz$.

Ex. Two similar paraboloidical vessels (Fig. 40.) are placed vertically above one another, and there is made an horizontal aperture near the vertex of the higher vessel, through which a fluid flows into the lower, and escapes over its sides. The fluid being uniformly supplied, it is required to determine when its surface will become stationary in the higher vessel.

Let $Q$ represent the constant influx in an unit of time, $a$ the distance of the vertex of the upper paraboloid below the level of the edges of the lower, $b$ the axis of the lower paraboloid, and $c$ the parameter of either vessel;

$$\therefore\ k = \pi cb - \pi ca, \qquad K = \pi c(z + a);$$

$$\therefore\ Q^2 = \frac{2\pi^2 c^2 gz}{\frac{1}{(b-a)^2} + \frac{1}{(z+a)^2}};$$

whence by reduction we obtain

$$z^3 + (2a - \alpha) z^2 + (a - 2\alpha) az + \alpha (b-a)^2 = 0;$$

$$\text{where} \qquad \alpha = \frac{Q^2}{2\pi^2 gc^2 (b-a)^2}.$$

*On the* variable *Motion of Fluids.*

156. To investigate the *variable* motion of fluids on the hypothesis of parallel sections in its most general form, let us return to the general equation of Art. 131. From whence we obtain

$$p=\int\{Xdx+Ydy+Zdz\}\mp\int D\{\phi dx+\phi' dy+\phi'' dz\}.$$

Let $ds$ represent the space described in the time $dt$, by the particle whose co-ordinates are $x$, $y$, $z$, and $v$ its velocity; then is $\frac{dv}{dt}$ the effective accelerating force upon it.

$$\therefore\ \phi=\frac{dv}{dt}\cdot\frac{dx}{ds},\quad \phi'=\frac{dv}{dt}\cdot\frac{dy}{ds},\quad \phi''=\frac{dv}{dt}\cdot\frac{dz}{ds};$$

$$\therefore\ p=\int D\{Xdx+Ydy+Zdz\}\mp\int D\left\{\frac{dx^2+dy^2+dz^2}{ds}\right\}\frac{dv}{dt}$$

$$=\int D\{Xdx+Ydy+Zdz\}\mp\int D\frac{dv}{dt}ds.$$

In the case in which the fluid is homogeneous, and the force that of gravity,

$$p=D\int gdz\mp D\int\frac{dv}{dt}ds\ldots\ldots\ldots\ldots..(A).$$

Now this integral must be taken throughout a line of particles extending to the surface of the fluid, *at the given instant* when the motion is to be determined.

The quantity $v$ is manifestly a function, as well of the *position* of the whole mass of fluid, as of the position of the given particle within it; or it is a function of $t$ as well as of the variables $x$, $y$, $z$. Now, $\int\frac{dv}{dt}ds$ is to be integrated at a *given time*, or *exclusively* with reference to the variables $x$, $y$, $z$. To do this, we must clearly, in

the first place, endeavour to express $\frac{dv}{dt}$ in terms of $x$, $y$, $z$, $t$, the variables on which it depends. With this view, let us have recourse to the hypothesis of parallel sections.

157. Let it be assumed that the motion is such that a plane being taken, at any instant, perpendicular to the motion of any particle of the fluid, all the other particles at the same instant passing through it, will move with the same velocities and in directions also perpendicular to it.

Let $\kappa$ be the area of any section taken as above, and $k$ that of any given section of the *vessel* or other containing surface through which the fluid is made to pass. Also, let $\upsilon$ and $v$ be the velocities of the fluid corresponding to the sections $\kappa$ and $k$.

$$\therefore \ \upsilon\kappa = vk; \qquad \therefore \ \upsilon = \frac{vk}{\kappa}.$$

Differentiating with respect to $t$,

$$\frac{d\upsilon}{dt} = \frac{k}{\kappa}\cdot\frac{dv}{dt} - \frac{vk}{\kappa^2}\cdot\frac{d\kappa}{dt} = \frac{k}{\kappa}\,\frac{dv}{dt} - \frac{vk}{\kappa^2}\cdot\frac{d\kappa}{ds}\cdot\frac{ds}{dt},$$

but $\frac{ds}{dt} = \upsilon = \frac{kv}{\kappa}$; $\qquad \therefore \ \frac{d\upsilon}{dt}\,ds = k\frac{dv}{dt}\,\frac{ds}{\kappa} - v^2k^2\cdot\frac{d\kappa}{\kappa^3}$.

Now $v$, and therefore $\frac{dv}{dt}$, is a function exclusively of the *time*, the *position* of the section $k$ being given. Since, therefore, the integration is to be performed considering $t$, and therefore $v$ and $\frac{dv}{dt}$, functions of that variable, constant;

$$\int\frac{d\upsilon}{dt}\,ds = k\frac{dv}{dt}\int\frac{ds}{\kappa} + \tfrac{1}{2}\frac{v^2k^2}{\kappa^2}.$$

$$\therefore \ p = gz \mp \left\{k\frac{dv}{dt}\int\frac{ds}{\kappa} + \tfrac{1}{2}\frac{v^2k^2}{\kappa^2}\right\} + C.$$

158. Let us take the case in which the motion tends to increase the co-ordinates. Let $K$ be the area of an *extreme* section or surface of the fluid, and let $z_1$ be the corresponding value of $z$, and $N$ the value of the integral $\int \frac{dz}{\kappa}$, taken from $z_1$ to $z$. Also, let $p_1$ be the value of the pressure on the surface.

$$\therefore\ p-p_1=g(z-z_1)-k\frac{dv}{dt}N-\tfrac{1}{2}v^2k^2\left(\frac{1}{\kappa^2}-\frac{1}{K^2}\right)\ldots\ldots(B).$$

*On the accelerated vertical Motion of a Fluid.*

159. Let us suppose the motion to be wholly vertical, and the fluid to be contained in a vessel, through an horizontal aperture in whose base it escapes.

Now, the section $k$ may be any whatever of the vessel. Let it coincide with that made by the surface of the fluid, or let $k=K$; also, let $V$ be the velocity at the surface.

$$\therefore\ p-p_1=g(z-z_1)-K\frac{dV}{dt}N-\tfrac{1}{2}V^2K^2\left(\frac{1}{\kappa^2}-\frac{1}{K^2}\right).$$

Now, at the orifice let $k$ be the value of $\kappa$, and $a$ the value of $z$: also, to simplify the notation, for $z_1$, write $z$. Then, since at the orifice $p_1=p$;

$$0=g(a-z)-KN\frac{dV}{dt}-\tfrac{1}{2}K^2V^2\left(\frac{1}{k^2}-\frac{1}{K^2}\right)\ldots\ldots\ldots(C).$$

Now, $\frac{dV}{dt}$ is the accelerating force on the surface of the fluid; also $z$ is the depth of the surface below a given fixed point;

$$\therefore\ \frac{dV}{dt}dz=VdV:$$

and multiplying by $dz$, we have

$$0=g(a-z)\,dz-KNV\,dV-\tfrac{1}{2}K^2\left(\frac{1}{k^2}-\frac{1}{K^2}\right)V^2dz\ldots\ldots(D)$$

$$\therefore\ dV^2+\left\{\frac{\frac{K^2}{k^2}-1}{KN}\right\}V^2dz=2g\left(\frac{a-z}{KN}\right)dz.$$

Now,

$$\left\{\frac{\frac{K^2}{k^2}-1}{KN}\right\}\quad\text{and}\quad\left(\frac{a-z}{KN}\right)$$

are functions of $z$: the above is, therefore, a linear equation.

$$\therefore\ V^2=\epsilon^{-\int\left(\frac{\frac{K^2}{k^2}-1}{KN}\right)dz}\cdot\int\left\{2g\left(\frac{a-z}{KN}\right)\epsilon^{\int\left(\frac{\frac{K^2}{k^2}-1}{KN}\right)dz}dz+C\right\}.$$

If $v$ represent the velocity at the orifice, $VK=vk$;

$$\therefore\ v^2=\frac{K^2}{k^2}\epsilon^{-\int\left(\frac{\frac{K^2}{k^2}-1}{KN}\right)dz}\cdot\int\left\{2g\left(\frac{a-z}{KN}\right)\epsilon^{\int\left(\frac{\frac{K^2}{k^2}-1}{KN}\right)dz}dz+C\right\}:$$

which equation involves a complete solution of the problem.

Suppose the vessel to be prismatic. In this case,

$$\int\frac{\frac{K^2}{k^2}-1}{NK}dz=\left(\frac{K^2}{k^2}-1\right)\int\frac{dz}{a-z}=\text{h. l. }(a-z)^{-\left(\frac{K^2}{k^2}-1\right)};$$

$$\therefore\ v^2=\frac{K^2}{k^2}(a-z)^{\left(\frac{K^2}{k^2}-1\right)}\cdot\left\{2g\int(a-z)^{-\left(\frac{K^2}{k^2}-1\right)}dz+C\right\},$$

$$v^2=\frac{2gK^2}{2k^2-K^2}\left\{a^{2-\frac{K^2}{k^2}}\cdot(a-z)^{\frac{K^2}{k^2}-1}-(a-z)\right\};$$

taking the integral from 0 to $z$.

If $K^2 = 2k^2$, we have $v^2 = \frac{0}{0}$, and the integral fails. We must in this case return to equation (*D*), by substitution, in which we obtain

$$g(a-z)\,dz - (a-z)\,V dV - \tfrac{1}{2}V^2 dz = 0;$$

$$\therefore\ 2g\frac{dz}{(a-z)} - \frac{(a-z)\,dV^2 - V^2 d(a-z)}{(a-z)^2} = 0;$$

$$\therefore\ 2g\ \text{h. l.}\left(\frac{a}{a-z}\right) - \frac{V^2}{a-z} = 0;$$

$$\therefore\ V^2 = 2g(a-z)\ \text{h. l.}\left(\frac{a}{a-z}\right);$$

$$\therefore\ v^2 = 2\frac{K^2}{k^2}g(a-z)\ \text{h. l.}\left(\frac{a}{a-z}\right).$$

The velocity is a maximum when

$$z = a\left\{1 - \left(\frac{k^2}{K^2-k^2}\right)^{\frac{k^2}{K^2-2k^2}}\right\}.$$

160. If the fluid be continually retained at the *same given altitude* $h$, the quantities $z$, $K$, $N$ are constant, and the equation (*C*) may be integrated *immediately*. By transposition we have

$$KN\frac{dV}{dt} = gh - \tfrac{1}{2}K^2V^2\left(\frac{1}{k^2} - \frac{1}{K^2}\right);$$

$$\therefore\ dt = \frac{KNdV}{gh - \frac{1}{2}K^2V^2\left(\frac{1}{k^2} - \frac{1}{K^2}\right)};$$

$$\therefore\ t = \frac{KN}{\sqrt{2gh\left(\frac{K^2}{k^2}-1\right)}}.\ \text{h. l.}\left\{\frac{\sqrt{2gh} + V\sqrt{\frac{K^2}{k^2}-1}}{\sqrt{2gh} - V\sqrt{\frac{K^2}{k^2}-1}}\right\} + C.$$

Now, $vk = VK$;

$$\therefore\ t = \frac{N}{\sqrt{2gh\left(\frac{1}{k^2} - \frac{1}{K^2}\right)}} \text{ h. l. } \left\{\frac{\sqrt{2gh} + v\sqrt{1 - \frac{k^2}{K^2}}}{\sqrt{2gh} - v\sqrt{1 - \frac{k^2}{K^2}}}\right\} + C.$$

When $t = 0$, $v = 0$; $\therefore\ C = 0$.

Let
$$\frac{\sqrt{2gh\left(\frac{1}{k^2} - \frac{1}{K^2}\right)}}{N} = \lambda;$$

$$\therefore\ \frac{\sqrt{2gh} + v\sqrt{1 - \frac{k^2}{K^2}}}{\sqrt{2gh} - v\sqrt{1 - \frac{k^2}{K^2}}} = \epsilon^{\lambda t};$$

whence
$$v = \left\{\frac{2gh}{1 - \frac{k^2}{K^2}}\right\}^{\frac{1}{2}} \cdot \left(\frac{\epsilon^{\lambda t} - 1}{\epsilon^{\lambda t} + 1}\right).$$

161. As $t$ increases, the quantity $\dfrac{\epsilon^{\lambda t} - 1}{\epsilon^{\lambda t} + 1}$ continually approaches to unity as its limit, and the value of $v$ to $\left\{\dfrac{2gh}{1 - \frac{k^2}{K^2}}\right\}^{\frac{1}{2}}$; which expression we have before shown to represent the velocity of the fluid, when the motion has at length become uniform.

It appears then, that on the hypothesis of parallel sections the velocity can never *strictly* become uniform. If, however, $k$ be not *very* nearly of the same magnitude with $K$, and the altitude $h$ be not exceeding small, the value of $\lambda$ is in all cases comparatively great, and the greater continually as the ratio $\dfrac{k}{K}$ is less, and the altitude $h$ greater. The fluid may, therefore, in the case we have supposed, be considered

to attain a velocity which is *very nearly* uniform after a *finite* or even an exceeding small interval of time.

162. In the case in which

$$k=K, \quad \lambda=0, \quad \text{and} \quad v=\frac{0}{0}.$$

The solution fails, therefore, and we must have recourse to the equation $(C)$; whence we obtain, making $k=K$,

$$ghdt=KNdV; \qquad \therefore\ ght=KNV.$$

Hence it appears, that *in this case* the velocity of the descending fluid *continually* increases with the time, and that an uniform motion is never attained.

163. If $k$ be greater than $K$, and

$$\lambda=\frac{\sqrt{2gh\left(\frac{1}{K^2}-\frac{1}{k^2}\right)}}{N},$$

$$v=\left\{\frac{2gh}{\frac{k^2}{K^2}-1}\right\}^{\frac{1}{2}}\cdot\left(\frac{\epsilon^{\lambda t\sqrt{-1}}-1}{\epsilon^{\lambda t\sqrt{-1}}+1}\right)\frac{1}{\sqrt{-1}};$$

$$\therefore\ v=\left\{\frac{2gh}{\frac{k^2}{K^2}-1}\right\}^{\frac{1}{2}}\tan\frac{\lambda t}{2}.$$

In this case, therefore, as in the preceding, the velocity increases with the time, until $\lambda t=\pi$, when it becomes infinite. It appears, then, that according to the conditions supposed, no *finite* influx can keep the fluid constantly at the same height in the vessel, during a time represented by the formula $t=\frac{\pi}{\lambda}$.

164. Let us suppose fluid to descend in a vessel formed by the combination of any number of smaller vessels. Let

$z$, $a_1$, $a_2$, &c. be respectively the distances between the surface and aperture in the higher vessel, and the distances between the apertures in the rest of the vessels; let $K_1$, $K_2$, $K_3$ represent the surfaces of the fluid in the vessels, and $k_1$, $k_2$, $k_3$, &c. the apertures. Also, let $p_1$, $p_2$, $p_3 \ldots p_n$ be respectively the pressures at the apertures. Then is $p_n$ the pressure of the atmosphere on the surface of the fluid in the upper vessel. Therefore (Art. 157.)

$$p_1 - p_n = gz - k\frac{dv}{dt}N_1 - \tfrac{1}{2}v^2k^2\left(\frac{1}{k_1^{\,2}} - \frac{1}{K_1^{\,2}}\right),$$

$$p_2 - p_1 = ga_1 - k\frac{dv}{dt}N_2 - \tfrac{1}{2}v^2k^2\left(\frac{1}{k_2^{\,2}} - \frac{1}{K_2^{\,2}}\right),$$

&c. = &c.

$$p_n - p_{n-1} = ga_{n-1} - k\frac{dv}{dt}N_n - \tfrac{1}{2}v^2k^2\left(\frac{1}{k_n^{\,2}} - \frac{1}{K_n^{\,2}}\right).$$

If, therefore, $a$ represent the height of the aperture of the highest above that of the lowest vessel, and $N$ the sum of the quantities $N_1$, $N_2$, &c.

$$0 = g(a+z) - k\,.\,\frac{dv}{dt}\,.\,N - \tfrac{1}{2}v^2k^2\,\Sigma\left(\frac{1}{k_n^{\,2}} - \frac{1}{K_n^{\,2}}\right).$$

Now, $k$ may be any section whatever of the vessel, $v$ being the velocity through it. As before, let it coincide with the surface $K$. Then, multiplying by $-dz$, we shall have

$$0 = -g(a+z)\,dz + kNV\,dV + \tfrac{1}{2}V^2K^2\,\Sigma\left(\frac{1}{k_n^{\,2}} - \frac{1}{K_n^{\,2}}\right)dz.$$

From this linear equation all the circumstances of the motion may be determined as before.

# CHAP. IV.

## ON HYDRAULICS, OR THE MOTION OF FLUIDS IN PIPES.

To determine the motion of an incompressible fluid in an exceedingly slender tube, the bore or transverse section of which is every where the same.

165. In all cases of fluid motion when the impressed force is gravity, we have

$$p = \int D\,(g dz - f ds).$$

Now this integral is to be taken throughout the fluid at *a given instant* of the motion. But, the section of the tube being every where the same, the motion of every particle of the fluid it contains, is at *any given instant* the same, or, the accelerating force $f$ on each particle is the same; and $\int f ds = fs$. On the supposition therefore that $D$ is constant,

$$p = D\,(gz - fs) + C \ldots\ldots\ldots\ldots (A).$$

Where $s$ is the distance, measured along the tube, from the extremity by which the fluid enters, to the point where the pressure is to be determined.

166. Suppose a fluid to enter a pipe, as above, from a reservoir whose surface is of infinite dimensions, as compared with the section of the pipe. It is required to determine the motion when it has become uniform. Let $z$ be the depth of any portion of the fluid in the tube beneath the surface of that in the reservoir. By equation ($A$),

$$p = Dgz + C,$$

since $f = 0$, the velocity being uniform. At the extremity where the tube communicates with the reservoir, let $z = z_,$, and $p = p_,$;

$$\therefore p_, = Dgz_, + C.$$

But (Art. 132.) $p_, = Dgz_, - \frac{1}{2} Dv^2 + p'$

$p'$ being the unit of atmospheric pressure on the surface, and $v$ the velocity at the orifice;

$$\therefore C = - \tfrac{1}{2} Dv^2 + p',$$

$$\text{and } p = Dgz - \tfrac{1}{2} Dv^2 + p'.$$

Hence at the extremity of the tube where the efflux takes place,

$$0 = Dgz - \tfrac{1}{2} Dv^2, \text{ and } v^2 = 2gz.$$

167. Suppose the fluid to be wholly contained in the tube. Let $s_,$ and $s_{,,}$ be, at any time, the distances of its *two surfaces* from either extremity of the tube, measured along it. And let $z_,$ and $z_{,,}$ be the corresponding depths of the surfaces. Then taking the integral ($A$) from one surface to the other, since at both $p$ equals the pressure of the atmosphere, we have

$$- g\,(z_{,,} - z_,) + f\,(s_{,,} - s_,) = 0;$$

$$\therefore f = g\left(\frac{z_{,,} - z_,}{s_{,,} - s_,}\right)^* \ldots\ldots\ldots\ldots\ldots\ldots(B),$$

and generally

* This equation results immediately, from the consideration that the motion is produced by the pressure of that portion of the fluid which is above the plane of the two surfaces. Now this pressure is equivalent to the weight of a fluid column of the same base and altitude. Therefore the effective moving force equals $g\,(z_{,,} - z_,)\,k$. And the mass moved is $(s_{,,} - s_,)\,k$; therefore the accelerating force

$$= g\left(\frac{z_{,,} - z_,}{s_{,,} - s_,}\right).$$

$$p=p'-g\,(z-z_{,})+g\left(\frac{z_{,,}-z_{,}}{s_{,,}-s_{,}}\right)(s-s_{,})\ldots\ldots\ldots(C).$$

$$\text{Now } v\,dv=f\,ds=g\left(\frac{z_{,,}-z_{,}}{s_{,,}-s_{,}}\right)ds_{,};$$

$$\therefore\ v^2=2g\int\left(\frac{z_{,,}-z_{,}}{s_{,,}-s_{,}}\right)ds_{,}\ldots\ldots\ldots\ldots\ldots(D).$$

If the *quantity of fluid* contained in the tube be constantly the same,

$$s_{,,}-s_{,}=\text{constant}=c;$$

$$\therefore\ v^2=\frac{2g}{c}\int(z_{,,}-z_{,})\,ds_{,}.$$

168. If the tube be incessantly supplied with fluid, so that the stream may be *continuous* from the point where it first entered it, $z_{,}$ is constant. And taking the origin of the co-ordinates at the point of influx; since $z_{,}=0$, and $s_{,}=0$,

$$v^2=2g\int\frac{z_{,,}ds_{,,}}{s_{,,}}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(E).$$

Ex. 1. Let the tube $ACA'$ (Fig. 50), curved at $C$, have its branches $AC$ and $A'C$ perfectly straight, and let them be inclined to the vertical at angles $\gamma$ and $\gamma'$.

Let $M$ and $N$ be any positions of the surfaces of a fluid *moving* in the tube, and let $P$, $Q$ be their respective positions when there is an equilibrium. And first let us suppose the *quantity* of fluid to remain the same throughout the motion; $\therefore$ $MN=PQ$, and taking away the common part $NP$, $MP=QN$. Now the altitude of $M$ above $N$

$$=z_{,,}-z_{,}=PM\cos\gamma+QN\cos\gamma'=PM(\cos\gamma+\cos\gamma').$$

Let $PM=x$; therefore, by equation $D$,

$$v^2 = -\frac{2g}{c}\int(\cos\gamma + \cos\gamma')\,x\,dx;$$

$$\therefore\ v^2 = \frac{2g}{c}(\cos\gamma + \cos\gamma')\,(x_{,}^2 - x^2),$$

$x_{,}$ representing the extent of the oscillation;

$$\therefore\ t = \left\{\frac{c}{2g(\cos\gamma + \cos\gamma')}\right\}^{\frac{1}{2}} \cos^{-1}\left(\frac{x}{x_{,}}\right).$$

The oscillations are therefore isochronous, and are performed in a time represented by the formula

$$\left\{\frac{c}{2g(\cos\gamma + \cos\gamma')}\right\}^{\frac{1}{2}} \cdot \pi.$$

Let the tube be inverted as in (Fig. 51). Here the accelerating force manifestly tends to increase the quantity $x$, and we have,

$$v^2 = \frac{2g}{c}\int(\cos\gamma + \cos\gamma')\,x\,dx$$

$$= \frac{2g}{c}(\cos\gamma + \cos\gamma')\,(x^2 - x_{,}^2);$$

$$\therefore\ t = \left\{\frac{c}{2g(\cos\gamma + \cos\gamma'}\right\}^{\frac{1}{2}} \cdot \int \frac{dx}{(x^2 - x_{,}^2)^{\frac{1}{2}}}$$

$$= \left\{\frac{c}{2g(\cos\gamma + \cos\gamma')}\right\}^{\frac{1}{2}} \text{h. l. } \frac{x + (x^2 - x_{,}^2)^{\frac{1}{2}}}{x_{,}}.$$

Next let us suppose the position of one of the surfaces of the fluid to remain unaltered. Let the *stationary* surface be in $A$, (Fig. 50), or (Fig. 51), and let the moving surface be in $N$.

Let $ACA' = c$, $ACN = s$; $\therefore\ NA' = \pm\,(c - s)$.

The sign $\pm$ being taken according as we take the position of (Fig. 50.), or the inverted position of (Fig. 52.) Hence we obtain

$$z_{\prime\prime} = \pm\ (c - s) \cos \gamma' ;$$

therefore by equation $E$,

$$v^2 = \pm\ 2g \cos \gamma' \int \left(\frac{c - s}{s}\right) ds$$

$$= \pm\ 2g \cos \gamma' \left\{ c \text{ h. l. } \frac{s}{s_{\prime}} - (s - s_{\prime}) \right\}.$$

The position of the surface $N$, at which the velocity $= 0$, or, in other words, the greatest distance to which the fluid can, on the hypothesis, be made to flow in the tube, is determined by the equation

$$c \text{ h. l. } \frac{s}{s_{\prime}} - (s - s_{\prime}) = 0.$$

Ex. 2. To determine the motion of fluid in a cycloidal tube.

Let us first suppose *a given* quantity of fluid to oscillate in the tube (Fig. 52.) Let $MN$ be the portion occupied by the fluid at any period of the motion.

Take $AP = \frac{1}{2} MN$, and let it be represented by $S$. Let $PM = s$, and $a =$ radius of generating circle;

$$\therefore\ 8a . \overline{AL} = \overline{AM}^2 = (S + s)^2,\ 8a . \overline{AK} = \overline{AN}^2 = (S - s)^2 ;$$

$$\therefore\ 8a . \overline{LK} = 4Ss ; \quad \therefore\ LK = \frac{Ss}{2a}.$$

Now by equation $D$,

$$v^2 = -\ 2g \int \frac{LK}{2S}\, ds = -\ \frac{g}{2a} \int s\, ds = \frac{g}{4a} (s_{\prime}^2 - s^2).$$

Whence we obtain

$$t = 2\sqrt{\frac{a}{g}} \int \frac{ds}{(s_{\prime}^2 - s^2)^{\frac{1}{2}}} = 2\sqrt{\frac{a}{g}} \cos^{-1}\left(\frac{s}{s_{\prime}}\right).$$

$$\text{When } s = 0,\ t = 2\sqrt{\frac{a}{g}} \cdot \frac{\pi}{2}.$$

The oscillations are therefore isochronous and are each performed in the time $2\pi\sqrt{\dfrac{a}{g}}$.

Suppose the tube (Fig. 52.) to be inverted. Then as before

$$LK = \frac{Ss}{2a};$$

$$\therefore v^2 = \frac{g}{S}\int \overline{LK}\, ds = \frac{g}{2a}\int s\, ds = \frac{g}{4a}(s^2 - s_{,}^2);$$

$$\therefore t = 2\sqrt{\frac{a}{g}}\int \frac{ds}{(s^2 - s_{,}^2)^{\frac{1}{2}}} = 2\sqrt{\frac{a}{g}}\ \text{h. l.}\ \frac{s + (s^2 - s_{,}^2)^{\frac{1}{2}}}{s_{,}}.$$

Suppose the tube to be kept continually full.

$$\text{Let } AM = \sigma; \quad \therefore AN = 2S - \sigma;$$

$$\therefore 8a\overline{LK} = \sigma^2 - (2S - \sigma)^2; \quad \therefore LK = \frac{S\sigma - S^2}{2a}.$$

Now (equation $E$),

$$v^2 = 2g\int \frac{\overline{LK}}{S}\, dS = \frac{g}{a}\int (\sigma - S)\, dS = \frac{g}{a}(\sigma S - \tfrac{1}{2}S^2);$$

$$\therefore t = \sqrt{\frac{2a}{g}}\int \frac{2\, dS}{\sqrt{2\sigma S - S^2}} = 2\sqrt{\frac{2a}{g}}\ \text{vers}^{-1}\left(\frac{S}{\sigma}\right).$$

The velocity vanishes when $S = 2\sigma$, or $AN = 3AM$, and after the time $2\pi\sqrt{\dfrac{2a}{g}}$.

Ex. 3. To find the time of the oscillation of a fluid in a tube in the form of a catenary (Fig. 53.)

Let $BM = s$, $AM = s_{,}$, $AC = b$, $z_{,}$ and $z_{,,}$ the depths of the surfaces of the fluid below the point $A$, $MN = c$;

$$\therefore z_{,} = b - \sqrt{a^2 + s^2}, \quad z_{,,} = b - \sqrt{a^2 + (c - s)^2}.$$

Therefore, (equation $D$),

$$v^2 = \frac{2g}{c} \int (z_{,,} - z_{,})\, ds_{,} = -\frac{2g}{c} \int \left\{ \sqrt{a^2 + s^2} - \sqrt{a^2 + (c - s)^2} \right\} ds,$$

$$v^2 = -\frac{g}{c} \left\{ a^2 \text{ h.l. } \{(c - s) + \sqrt{a^2 + (c - s)^2}\} \{s + \sqrt{a^2 + s^2}\} \right.$$

$$\left. + (c - s)\sqrt{a^2 + (c - s)^2} + s\sqrt{a^2 + s^2} \right\} + C.$$

169. To determine the motion of a fluid in an *unequal* tube $BC$, (Fig. 54.), the transverse sections $k_{,}$ and $k_{,,}$ of whose branches $AB$ and $AC$ are the same throughout each branch.

Let $f_{,}$ and $f_{,,}$ represent the effective accelerating forces, and $v_{,}$, $v_{,,}$ the velocities of the fluid in the branches $AB$ and $AC$ of the tube respectively, and let $M$ and $N$ be, at any given time, the positions of the surfaces. Let $AM = s_{,}$, $AN = s_{,,}$, $AP = S_{,}$, $AQ = S_{,,}$.

Integrating equation ($A$) throughout the fluid, we obtain

$$-f_{,}s_{,} - f_{,,}s_{,,} + g(z_{,} - z_{,,}) = 0.$$

Now, supposing every particle in each transverse section to move with the same velocity, we have

$$v_{,}k_{,} = v_{,,}k_{,,}; \quad \therefore \frac{dv_{,}}{dt} k_{,} = \frac{dv_{,,}}{dt} k_{,,}; \quad \therefore f_{,}k_{,} = f_{,,}k_{,,}.$$

Whence, eliminating in the preceding equation, we obtain

$$f_{,} = gk_{,,} \cdot \frac{z_{,} - z_{,,}}{s_{,}k_{,,} + s_{,,}k_{,}}; \quad \therefore v_{,}^2 = 2gk_{,,} \int \frac{z_{,} - z_{,,}}{s_{,}k_{,,} + s_{,,}k_{,}}\, ds_{,}.$$

Ex. 1. Suppose *a given quantity of fluid*, $c$, to oscillate in a tube whose arms are straight, and let $PAQ$ be the position

of equilibrium. Let $PM=\sigma$, inclinations of $PB$ and $QC$ to the vertical $=\gamma_{,}$ and $\gamma_{,,}$.

$$\overline{AM}\,.\,k_{,}+\overline{AN}\,.\,k_{,,}=\overline{AP}\,.\,k_{,}+\overline{AQ}\,.\,k_{,,};\quad \overline{PM}\,.\,k_{,}=\overline{QN}\,.\,k_{,,}.$$

$$\text{Now } z_{,}-z_{,,}=\overline{PM}\cos\gamma_{,}+\overline{QN}\cos\gamma_{,,}=\sigma\cos\gamma_{,}+\frac{k_{,}}{k_{,,}}\sigma\cos\gamma_{,,},$$

$$\text{also } s_{,}k_{,,}+s_{,,}k_{,}=(S_{,}+\sigma)\,k_{,,}+\left(S_{,,}-\frac{k_{,}\sigma}{k_{,,}}\right)k_{,},$$

whence we obtain by substitution,

$$v^2=2gk_{,,}\,(k_{,,}\cos\gamma_{,}+k_{,}\cos\gamma_{,,})\int\frac{-\sigma d\sigma}{S_{,}k^2_{,,}+S_{,,}k_{,,}k_{,}+(k_{,,}^2-k_{,}^2)\,\sigma}.$$

$$\text{Let } \frac{S_{,}k_{,,}^2+S_{,,}k_{,,}k_{,}}{k_{,}^2-k_{,,}^2}=A;$$

$$\therefore\ v^2=\frac{2gk_{,,}\,(k_{,,}\cos\gamma_{,}+k_{,}\cos\gamma_{,,})}{k_{,}^2-k_{,,}^2}\left\{(\sigma-\sigma_{,})+A\,\text{h.l.}\frac{A-\sigma}{A-\sigma_{,}}\right\},$$

$\sigma_{,}$ being the value of $\sigma$ when $v=0$.

If the oscillations be *exceeding small;* neglecting the powers of $\sigma$ and $\sigma_{,}$ above their squares, we have

$$v^2=g\,\frac{k_{,,}\,(k_{,,}\cos\gamma_{,}+k_{,}\cos\gamma_{,,})}{S_{,}k_{,,}^2+S_{,,}k_{,,}k_{,}}\,(\sigma_{,}^2-\sigma^2).$$

Whence it appears that the *small* oscillations are *isochronous*, and that they are performed in the time,

$$\pi\sqrt{\frac{S_{,}k_{,,}+S_{,,}k_{,}}{g\,(k_{,,}\cos\gamma_{,}+k_{,}\cos\gamma_{,,})}}.$$

If the surface $Q$ of the fluid be kept continually at the same height we shall obtain from equation $E$,

$$v^2=V^2+2g\cos\gamma_{,}\left\{\left(\frac{S_{,}k_{,,}+S_{,,}k_{,}}{k_{,,}}\right)\text{h.l.}\left(\frac{S_{,,}k_{,}+S_{,}k_{,,}}{S_{,,}k_{,}}\right)-s_{,}\right\};$$

where $V$ is the velocity, when $s_{,}=0$.

170. If the bore of the tube be not the same throughout, the quantity $f$ becomes a function both *of the position and the time*, and we must have recourse to the hypothesis of parallel sections.

By Article (156), we have

$$p = gz - k\frac{dv}{dt}\int\frac{ds}{\kappa} - \frac{1}{2}\frac{k^2v^2}{\kappa^2} + C.$$

Taking the integral from one surface to the other, and representing $\int\frac{ds}{\kappa}$ by $N$, we have, since the pressure is the same on either surface,

$$0 = g(z_{,,} - z_{,}) - k\frac{dv}{dt}\cdot N - \frac{1}{2}k^2v^2\left(\frac{1}{K_{,}^2} - \frac{1}{K_{,,}^2}\right);$$

where $K_{,}$ and $K_{,,}$ are the sections of the tube at the surfaces of the fluid. Now $k$ is any section whatever of the tube; let it coincide with that surface $K_{,}$ of the fluid from which the integral $N$ is taken. And let $V$ be the corresponding velocity;

$$\therefore\ 0 = g(z_{,,} - z_{,}) - K_{,}N\frac{dV}{dt} - \frac{1}{2}V^2\left(1 - \frac{K_{,}^2}{K_{,,}^2}\right):$$

$$\therefore\ 0 = g(z_{,,} - z_{,})\,ds - K_{,}N\,VdV - \frac{1}{2}V^2\left(1 - \frac{K_{,}^2}{K_{,,}^2}\right)ds_{,}.$$

Now the dimensions of the tube being given, $K_{,}$ is given in terms of $z_{,}$, also the quantity of fluid being given, $K_{,,}$ and $N$ are given in terms of $z_{,}$ and $z_{,,}$, and $z_{,,}$ is given in terms of $z_{,}$, and $z_{,}$ in terms of $s_{,}$. The quantities $K_{,}$, $K_{,,}$, $N$, $z_{,}$, $z_{,,}$, $s_{,}$ are therefore all given in terms of one of them $s_{,}$. The equation is therefore linear and may be integrated as before.

# CHAP. V.

## ON THE RESISTANCE OF FLUIDS.

171. LET a plane surface $M$ be supposed to move with an uniform velocity $v$ in a fluid of infinite dimensions. Suppose also the motion of the *fluid* to be uniform and in a direction opposite to that of the body, and let its velocity be $V$.

Let $f$ be the effective accelerating force generated or destroyed in any particle $\mu$ of the fluid, by the reaction of the plane, and estimated in a direction perpendicular to its surface. Also let $R$ represent the pressure on the plane. Then is $-R$ the resistance or force *impressed* on the fluid, and $\Sigma f\mu$ the whole *effective* force estimated in a direction perpendicular to the plane. And therefore by d'Alembert's principle,

$$\Sigma f\mu - R = 0 \ldots\ldots\ldots\ldots\ldots\ldots (A).$$

The integral $\Sigma f\mu$ is to be taken with regard to the whole fluid mass at a given instant of time. Now since the velocity of the stream, and that of the body are constant, the disturbance is uniform, or the accelerating force $(f)$, generated or destroyed, in every particle, similarly situated with regard to the plane, is the same in every part of its course. Considering therefore the line of particles which occupies the path of any particle $(\mu)$, at present in contact with the surface of the plane, it appears that $f$ is, with regard to each of the particles which compose that line, the same as it was with regard to $\mu$, when it occupied the same relative position. Now with reference to this line of particles, if $ds$ represent an element of its length and $\Delta M$ that element of the plane $M$ which forms its base, we have,

$$\Sigma f\mu = \int D \,.\, f \Delta M \,.\, ds = D \,.\, \Delta M \int f ds.$$

But $\int f ds$, taken with respect to the different particles of the column at any *given instant*, is the same with $\int f ds$ taken with respect to the single particle $\mu$, during the *different successive instants* of its motion.

Calling $v'$ the whole velocity generated or destroyed in the particle $\mu$, estimated in a direction perpendicular to the plane, we have therefore $\int f ds = \frac{1}{2} v'^2$. And, as far as it regards the line of particles in question,

$$\Sigma f \mu = \tfrac{1}{2} D \,.\, \Delta M \,.\, v'^2;$$

the integral being taken from those particles among which no disturbance is produced, or $v' = 0$, to those in immediate contact with the plane.

On the whole therefore

$$R = \Sigma f \mu = \tfrac{1}{2} D \Sigma v'^2 \,.\, \Delta M.$$

Now if the direction of the motion of the plane, and that of the stream be inclined to a perpendicular to the surface of the plane at an angle $\phi$; the velocities of the plane and fluid, resolved in that direction are respectively $V \cos \phi$, and $v \cos \phi$. Also the velocity perpendicular to the plane is destroyed in that fluid which is immediately contiguous to it, and a velocity equal to its own is generated in the opposite direction. On the whole therefore the velocity lost in a direction perpendicular to the plane is $(V \mp v) \cos \phi$.

$$\therefore \; R = \tfrac{1}{2} D \Sigma \, (V \mp v)^2 \cos^2 \phi \, \Delta M.$$

And if we conceive the velocity of every portion of the fluid at present in contact with the plane to have been the same before the disturbance,

$$R = \tfrac{1}{2} DM \, (V \mp v)^2 \cos^2 \phi \ldots\ldots\ldots\ldots\ldots\ldots (B).$$

If the plane be *perpendicular* to the direction of the stream, $\phi = 0$, and the general expression becomes

$$R = \frac{1}{2} DM (V \mp v)^2 * \ldots\ldots\ldots\ldots\ldots\ldots\ldots (C).$$

172. The above is the resistance to motion in a direction perpendicular to the plane. Now this direction makes, by supposition, an angle $\phi$ with the direction of the stream. Resolved in this direction, that is, in that of the motion of the plane, the resistance becomes therefore

$$R \cos\phi = \frac{1}{2} DM (V \mp v)^2 \cos^3\phi \ldots\ldots\ldots (D).$$

And resolved in a direction perpendicular to the stream; it is,

$$R \sin\phi = \frac{1}{2} DM (V \mp v)^2 \cos^2\phi \,.\, \sin\phi \ldots\ldots (E).$$

173. This last quantity is a maximum when $\cos^2\phi \sin\phi$ is a maximum, or when

$$-2\sin^2\phi + \cos^2\phi = 0, \text{ or } \tan\phi = \frac{1}{\sqrt{2}}.$$

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* The theory of resistances may be deduced from the *general theorem* (Art. 132.) as follows:

Suppose a fluid to impinge with an uniform velocity on a plane at rest.

Now, considering the *line* of particles which is in the path of a given particle of the impinging fluid, taking the plane $xy$ beneath it, and calling $p$, $z$, $v$ the pressure, altitude, and velocity at the point where the *disturbance* of the given particle by the plane commences; and $p_{,}$, $z_{,}$, $v_{,}$ those at the point where it comes in contact with the plane; since the motion is in the direction in which the pressure is estimated, or tends to increase the co-ordinates, we have (Art. 132.)

$$p - p_{,} = Dg(z - z_{,}) - \tfrac{1}{2} D (v^2 - v_{,}^2).$$

Now, if we suppose the pressure at the *posterior* surface of the plane not to be affected by the disturbance of the fluid, the unit of pressure on the corresponding point of that surface will be represented also by $p$; $\therefore p_{,} - p$ is the *difference* of the units of pressure at corresponding points of the two opposite surfaces of the plane;

$$\therefore \Sigma(p_{,} - p)\mu = Dg\Sigma(z_{,} - z)\mu + \tfrac{1}{2} D\Sigma(v^2 - v_{,}^2)\mu$$

is the *whole* pressure tending to produce motion in the plane.

*If*, therefore, $z = z_{,}$, and $v_{,} = 0$, resistance $= \frac{1}{2} D\Sigma v^2 \mu$. (See Appendix D.)

The angle thus determined is clearly that at which the rudder of a boat must be inclined to the stream, to produce the greatest possible effect in turning it.

174. If a body be symmetrical with regard to a certain vertical plane, and move in a direction parallel to that plane; the resistances on the two symmetrical portions of it, resolved in directions perpendicular to its motion, will manifestly be equal and opposite. The whole *effective* resistance is therefore *in* the direction of the motion of the body.

Ex. Let a symmetrical wedge be supposed to move in the direction of its axis. Let either face be represented by $M$, and let $\theta$ be the inclination of the faces to one another. Then is $\frac{\theta}{2}$ the inclination of either face to the direction of the stream: therefore by equation $(D)$,

$$\text{resistance} = DM\,(V \mp v)^2 \sin^3 \frac{\theta}{2}.$$

175. To find the resistance on a solid of revolution moving in the direction of its axis.

Suppose the surface to be made up of elementary planes $PQ$, (Fig. 55.) formed by sections made *through* the axis $AB$, and *perpendicular* to it. Let $PT$ be a tangent to the generating curve in $P$; then is $PTA$ the inclination of the plane $PQ$ to the direction of the motion. And the resistance on $PQ$ in that direction

$$= \tfrac{1}{2} D\,(V \mp v)^2 . PQ . \cos^3 \left(\frac{\pi}{2} - PTA\right)$$

$$= \tfrac{1}{2} D . PQ . (V \mp v)^2 \sin^3 PTA.$$

Now the angle $PTA$ is the same for every plane similarly taken in the annulus; therefore on the whole the resistance on the annulus $KQ = \frac{1}{2} D . KQ\,(V \mp v)^2 \sin^3 PTA$. But if $x$ and $y$ be co-ordinates of the point $P$, in the generating curve

$$\sin PTA = \frac{dy}{ds}, \text{ and } KQ = 2\pi y ds,$$

resistance on annulus $KQ = \pi D\,(V \mp v)^2\, y \left(\frac{dy}{ds}\right)^3 ds$

And whole resistance $= \pi D\,(V \mp v)^2 \int y \left(\frac{dy}{ds}\right)^2 dy,$

$$\ldots\ldots\ldots\ldots\ldots\ldots = \pi D\,(V \mp v)^2 \int y \left\{1 + \left(\frac{dy}{dx}\right)^{-2}\right\}^{-1} dy \;(F).$$

The above is the *resistance* or whole impressed force generated by the fluid in a direction opposite to the bodies' motion. The effective *retarding force* *

$$= \frac{\text{impressed force}}{\text{mass}} = \frac{D}{D'}(V \mp v)^2 \frac{\int y \left\{1 + \left(\frac{dy}{dx}\right)^{-2}\right\}^{-1} dy}{\int y^2 dx} \ldots (G).$$

Ex. 1. To find the resistance and retarding force on a sphere.

Here the equation to the generating curve is

$$a - x = (a^2 - y^2)^{\frac{1}{2}};$$

$$\therefore\; -\frac{dx}{dy} = -\frac{y}{(a^2 - y^2)^{\frac{1}{2}}}; \quad \therefore \left(\frac{dy}{dx}\right)^{-2} + 1 = \frac{a^2}{a^2 - y^2};$$

therefore resistance $= \dfrac{\pi D\,(V \mp v)^2}{a^2} \int (a^2 - y^2)\, y\, dy$

$$\ldots\ldots\ldots\ldots\ldots\ldots = \frac{\pi D (V \mp v)^2}{a^2} \left\{\tfrac{1}{2} a^4 - \tfrac{1}{4} a^4\right\} = \frac{\pi D (V \mp v)^2 a^2}{4};$$

taking the integral from $y = 0$, to $y = a$. Now the volume of the sphere is $\frac{4}{3}\pi a^3$. If therefore $D'$ represent its density, and the fluid be at rest, the retarding force $= \dfrac{3Dv^2}{16D'a} = \dfrac{3\sigma v^2}{16a}$; where $\sigma$ is the ratio of the specific gravities of the solid and fluid.

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* This term is used in opposition to *accelerating force.*

Ex. 2. To find the resistance on a spheroid.

$$\text{Here}\quad a - x = \frac{a}{b}(b^2 - y^2)^{\frac{1}{2}};$$

$$\therefore\ -\frac{dx}{dy} = \frac{a}{b}\,\frac{-y}{(b^2 - y^2)^{\frac{1}{2}}};$$

$$\therefore\ 1 + \left(\frac{dy}{dx}\right)^{-2} = \frac{b^4 - b^2y^2 + a^2y^2}{b^2(b^2 - y^2)} = \frac{b^4 + a^2e^2y^2}{b^2(b^2 - y^2)};$$

$$\therefore\ \text{resistance} = \pi D (V \mp v)^2 b^2 \int \frac{(b^2 - y^2)\,y\,dy}{b^4 + a^2e^2y^2}$$

$$\ldots\ldots\ldots\ldots = \frac{\pi D (V \mp v)^2 b^2}{a^2e^2} \int \frac{(b^2a^2e^2 - a^2e^2y^2 + b^4 - b^4)\,y\,dy}{b^4 + a^2e^2y^2}$$

$$\ldots\ldots\ldots\ldots = \frac{\pi D (V \mp v)^2 b^2}{a^2e^2} \left\{ \int \frac{(b^2a^2e^2 + b^4)\,y\,dy}{b^4 + a^2e^2y^2} - \int y\,dy \right\}$$

$$\ldots\ldots\ldots\ldots = \frac{\pi D (V \mp v)^2 b^4}{2a^2e^2} \left\{ \left(\frac{b^2}{a^2e^2} + 1\right) \text{h.l.}\left(1 + \frac{a^2e^2}{b^2}\right) - 1 \right\};$$

taking the integral from $y = 0$, to $y = b$. The mass of the spheroid is $\frac{4}{3}ab^2\pi D'$; therefore the retarding force

$$= \frac{3}{8}\,\frac{(V \mp v)^2 b^2 \sigma}{a^3e^2} \left\{ \left(\frac{b^2}{a^2e^2} + 1\right) \text{h.l.}\left(1 + \frac{a^2e^2}{b^2}\right) - 1 \right\}.$$

Ex. 3. To find the resistance on a solid generated by the revolution of a cycloid about its axis, and moving in the direction of that axis. By the nature of the cycloid

$$x = a(1 - \cos\theta),\quad y = a(\theta + \sin\theta);$$

$$\therefore\ \frac{dx}{dy} = \frac{\sin\theta}{1 + \cos\theta} = \tan\frac{\theta}{2};\quad \therefore\ 1 + \left(\frac{dy}{dx}\right)^{-2} = \sec^2\frac{\theta}{2}.$$

$$\text{Also}\ y\,dy = a^2(\theta + \sin\theta)(1 + \cos\theta)\,d\theta$$

$$= 2a^2(\theta + \sin\theta)\cos^2\frac{\theta}{2}\,d\theta;$$

therefore resistance $= 2\pi D\,(V \mp v)^2\, a^2 \int (\theta + \sin\theta)\cos^4\frac{\theta}{2}\, d\theta.$

$$\text{Now } \cos^4\frac{\theta}{2} = \frac{1}{8}\left\{\cos 2\theta + 4\cos\theta + 3\right\};$$

$$\therefore \int \cos^4\frac{\theta}{2}\,.\,\theta\,.\,d\theta = \frac{1}{8}\left\{\tfrac{1}{4}(2\theta\sin 2\theta + \cos 2\theta) + 4(\theta\sin\theta + \cos\theta) + \frac{3}{2}\theta^2\right\}.$$

$$\text{Also} \int \sin\theta\cos^4\frac{\theta}{2}\, d\theta = -4\int \cos^5\frac{\theta}{2}\, d\cos\frac{\theta}{2} = -\frac{2}{3}\cos^6\frac{\theta}{2};$$

therefore resistance $= 2\pi D\,(V \mp v)^2$

$$a^2\left\{\frac{\theta\sin 2\theta}{16} + \frac{\cos 2\theta}{32} + \frac{\theta\sin\theta}{2} + \frac{\cos\theta}{2} + \frac{3}{16}\theta^2 - \frac{2}{3}\cos^6\frac{\theta}{2}\right\} + C.$$

And taking the integral from $\theta = 0$, to $\theta = \pi$,

$$\text{resistance} = \pi D\,(V \mp v)^2\,.\,a^2\,.\left\{\frac{3\pi^2}{8} + \frac{1}{3}\right\}.$$

176. To find the resistance on any symmetrical body moving in the direction of its axis.

Let $u = 0$ be the equation to the surface of the body, the axis about which it is symmetrical being taken for the axis of $x$.

Then the line of the inclination of any elementary portion of the surface, whose co-ordinates are $x$, $y$, $z$, to the direction of motion is

$$\frac{\dfrac{du}{dx}}{\left\{\left(\dfrac{du}{dx}\right)^2 + \left(\dfrac{du}{dy}\right)^2 + \left(\dfrac{du}{dz}\right)^2\right\}^{\frac{1}{2}}}.$$

Now the area of the element is

$$\frac{\left\{\left(\frac{du}{dx}\right)^2 + \left(\frac{du}{dy}\right)^2 + \left(\frac{du}{dz}\right)^2\right\}^{\frac{1}{2}}}{\left(\frac{du}{dx}\right)} dy dz;$$

$$\therefore \text{ resistance} = \tfrac{1}{2} D (V \mp v)^2 \iint \frac{\left(\frac{du}{dx}\right)^2 dy\, dz}{\left(\frac{du}{dx}\right)^2 + \left(\frac{du}{dy}\right)^2 + \left(\frac{du}{dz}\right)^2}.$$

177. *On the rectilinear motion of a body in a resisting medium.*

Suppose the medium at rest, and let $kv^2$* represent its retarding force on the motion of the body. Let the motion take place wholly in the direction of the axis about which the body is symmetrical, and let the fluid be homogeneous. So that the quantity $k$, dependent on the mass, the ratio of the densities and *the surface on which the resistance takes place;* may be the same throughout the motion.

Let $x$ be the distance, at any time, from the point of projection, and $\pm P$ the force accelerating or retarding the motion of the body.

The whole accelerating or retarding force impressed on the body at any period of its motion is $\pm P - kv^2$;

$$\therefore vdv = \pm Pdx - kv^2dx \ldots\ldots\ldots\ldots\ldots(1);$$

$$\therefore dv^2 + 2kv^2dx = \pm 2Pdx;$$

$$\therefore v^2 = \epsilon^{-2kx}\left\{\pm 2\int P\epsilon^{2kx}dx + C\right\}\ldots\ldots(2).$$

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* The retarding force has been shown to vary as the square of the velocity, in *that case only* in which the motion of the body is uniform. *The theorem cannot be extended to the case of variable motion except as an approximation.* It is in this sense that it is here given.

178. In the case which $P$ is constant,

$$v^2 = \epsilon^{-2kx}\left\{\pm 2P\int \epsilon^{2kx}\,dx + C\right\} = \epsilon^{-2kx}\left\{\pm \frac{P}{k}(\epsilon^{2kx}-1) + v_{,}^{2}\right\},$$

where $v_{,}$ is the velocity of projection.

179. When $x = \infty^{\text{ty}}$, $v^2 = \pm\frac{P}{k}$.

Taking the positive sign we find that $\sqrt{\frac{P}{k}}$ is a quantity to which the velocity continually approximates as the distance increases, but which, in any finite distance, it never actually attains. This velocity is called the terminal velocity.

180. If the negative sign be taken, the terminal value of $v$ becomes impossible. The motion in this case therefore ceases at a *finite* distance from the point of projection. To find this distance, let $v = 0$;

$$\therefore\ -\frac{P}{k}(1-\epsilon^{-2kx}) + v_{,}^{2}\epsilon^{-2kx} = 0,$$

whence we obtain $x = \frac{1}{2k}\,\text{h. l.}\left(\frac{kv_{,}^{2}}{P} + 1\right)$.

181. $$t = \int\frac{dx}{v} = \int\frac{dx}{\left\{\pm\frac{P}{k} - \left(\pm\frac{P}{k} - v_{,}^{2}\right)\epsilon^{-2kx}\right\}^{\frac{1}{2}}}$$

$$= \frac{1}{\sqrt{Pk}}\int\frac{d\epsilon^{kx}}{\left\{\left(\mp 1 + \frac{v_{,}^{2}k}{P}\right) \pm \epsilon^{2kx}\right\}}.$$

The integral of this expression is a logarithmic or circular function according as the upper or lower sign is taken; that is, according as the force is in the direction of the motion, or opposed to it. In the first case we have, taking the integral from 0 to $x$,

$$t = \frac{1}{\sqrt{kP}} \text{ h. l. } \left\{ \frac{v_{,} \sqrt{\frac{k}{P}} - 1}{\sqrt{\epsilon^{2kx} + \frac{v_{,}^2 k}{P} - 1} - \epsilon^{kx}} \right\}.$$

In the seconnd case,

$$t = \frac{1}{\sqrt{Pk}} \left\{ \cos^{-1} \frac{1}{\left(1 + \frac{v_{,}^2 k}{P}\right)^{\frac{1}{2}}} - \cos^{-1} \frac{\epsilon^{kx}}{\left(1 + \frac{v_{,}^2 k}{P}\right)^{\frac{1}{2}}} \right\}.$$

Substituting for $x$ its maximum value, we obtain for the time in which the greatest distance is attained,

$$t = \frac{1}{\sqrt{Pk}} \cos^{-1} \frac{1}{\left(1 + \frac{v_{,}^2 k}{P}\right)^{\frac{1}{2}}}.$$

$$\text{When } v_{,} = \infty^{\text{ty}}, \quad t = \frac{\pi}{2\sqrt{Pk}}.$$

Which quantity may be considered as a limit never exceeded by the time of flight.

182. The distance $x$ may immediately be found in terms of the velocity $v$, (and conversely), when $P$ is constant, by equation (1), from which we obtain,

$$dx = \frac{\frac{1}{2} dv^2}{\pm P - kv^2}; \quad \therefore \ x = \frac{1}{2k} \text{ h. l. } \left( \frac{\pm P - kv_{,}^2}{\pm P - kv^2} \right).$$

183. *On the small vertical oscillations of floating bodies as affected by resistance.*

That portion of the accelerating force, on bodies oscillating vertically, which arises from the pressure of the fluid (independently of the resistance), varies as the distance of the plane of equilibrium from the surface of the fluid, as long as that distance is small. Call it $x$. Let $v$ be the corresponding velocity, and suppose the *variation* of the resisting surface to be small as compared with the whole surface, so that $k$ may be considered as remaining constant during the motion.

Let $Mx$ equal the accelerating force produced by the pressure of the fluid;

$$\therefore\ P = -Mx.$$

$$\text{and } v^2 = \epsilon^{-2kx}\left\{-2M\int x\epsilon^{2kx}dx + C\right\}$$

$$= \epsilon^{-2kx}\left\{-\frac{M}{k}\left(\epsilon^{2kx}x - \frac{\epsilon^{2kx}}{2k}\right) + C\right\}$$

$$= -\frac{M}{2k^2}\left\{(2kx-1)-(2kx_,-1)\,\epsilon^{2k(x_,-x)}\right\}.$$

Taking the integral from $x_,$ to $x$. Now neglecting terms of above three dimensions in $x$ and $x_,$,

$$(2kx_,-1)\,\epsilon^{2k(x_,-x)} = (2kx-1)+2k^2(x_,^2-x^2)+4k^3x_,(x_,-x)^2;$$

$$\therefore\ v^2 = M\left\{(x_,^2-x^2)+2kx_,(x_,-x)^2\right\}.$$

Whence we obtain by reduction

$$v^2 = M(1-2kx_,)\left\{\frac{x_,^2}{(1-2kx_,)^2} - \left(x+\frac{2kx_,^2}{1-2kx_,}\right)^2\right\};$$

$$\therefore\ t = \frac{\cos^{-1}2kx_, - \cos^{-1}\left\{\dfrac{x(1-2kx_,)+2kx_,^2}{x_,}\right\}}{\sqrt{M(1-2kx_,)}}.$$

For the whole oscillation,

$$t = \frac{\cos^{-1}2kx_,}{\sqrt{M(1-2kx_,)}}.$$

184. *To determine the vertical motion of a body through the atmosphere, when acted on by no forces, but simply impelled by the velocity of its projection.*

Let $\dfrac{\int y\left\{1+\left(\dfrac{dy}{dx}\right)^{-2}\right\}^{-1}dy}{\int y^2dx} = k.$ Therefore

retarding force $= \frac{D}{D'} v^2 k$. $D$ being the density of the air, and $D'$ that of the body. Now if we consider gravity constant, we have (Art. 121.) $D = D_{,}\, \epsilon^{-\frac{z}{h}}$, where $D_{,}$ is the density of the atmosphere, when $z = 0$. If therefore $\sigma = \frac{D_{,}}{D'}$; retarding force $= k\sigma v^2 \epsilon^{-\frac{z}{h}}$;

$$\therefore\ -v\,dv = k\sigma v^2 \epsilon^{-\frac{z}{h}} dz; \quad \therefore\ \frac{dv^2}{v^2} = -2k\sigma\epsilon^{-\frac{z}{h}} dz;$$

$$\therefore\ \text{h.l.}\ \frac{v^2}{v_{,}^2} = 2kh\sigma\,(\epsilon^{-\frac{z}{h}} - 1); \quad \therefore\ \text{h.l.}\ \frac{v}{v_{,}} = kh\sigma\,(\epsilon^{-\frac{z}{h}} - 1);$$

$$\therefore\ v = v_{,}\epsilon^{kh\sigma(\epsilon^{-\frac{z}{h}}-1)}; \quad \therefore\ t = \frac{1}{v_{,}}\int \epsilon^{kh\sigma(1-\epsilon^{-\frac{z}{h}})} dz.$$

**185.** The above theory is applicable to the motion of rays of light incident nearly perpendicularly on the atmosphere.

If we suppose $v$ and $v_{,}$ to be the velocities in two consecutive media, $\phi$, $\phi_{,}$ the angles of incidence and refraction at their common surface, and $D$, and $D'$ their densities, we have (Poisson, Art. 307.)

$$\frac{v_{,}}{v} = \frac{\sin\phi}{\sin\phi_{,}}; \quad \therefore\ \frac{v_{,} - v}{v_{,} + v} = \frac{\tan\frac{1}{2}(\phi - \phi_{,})}{\tan\frac{1}{2}(\phi + \phi_{,})}.$$

But since $v$ and $v_{,}$ are exceeding great, and $v_{,} - v$ finite, $v + v_{,} = 2v$. Also $\phi - \phi_{,}$ is exceeding small; therefore $\phi + \phi_{,} = 2\phi$;

$$\therefore\ \frac{v_{,} - v}{2v} = \frac{\tan\frac{1}{2}(\phi - \phi_{,})}{\tan\phi}; \quad \therefore\ \phi - \phi_{,} = \frac{v_{,} - v}{v}\tan\phi, \text{ nearly.}$$

Now by the last article,

$$v = v_{,}\epsilon^{kh\sigma(\epsilon^{-\frac{z}{h}}-1)} = v_{,}\epsilon^{\frac{kh}{D'}(D - D_{,})} = v_{,}\left\{1 + \frac{kh}{D'}(D - D_{,})\right\},$$

very nearly; since $(D - D_{\prime})$ is exceeding small.

$$\therefore \frac{v - v_{\prime}}{v_{\prime}} = \frac{kh}{D'}(D - D_{\prime}),$$

$$\text{and } \phi_{\prime} - \phi = \frac{kh}{D'}(D - D_{\prime}) \tan\phi.$$

Which is precisely the result given by observation*.

186. We have supposed the same theory of resistance to obtain in incompressible and elastic fluids. This is however by no means the case. The integral $\int D . \Delta M . f ds$ is manifestly to be taken in elastic fluids; regard being had to the variation of density produced by the resistance. This variation cannot be determined on any principles hitherto established. If, however, only that portion of the fluid which is in immediate contact with the body be considered, and its density be supposed uniform; then calling $D$ the density before the motion, and $R$, at any time, the resistance. The density will be represented by

$$D + C . R.$$

Since the increment of density varies as $R$.

$$\therefore R = M(D + CR)\int f ds = \tfrac{1}{2} M (D + C . R)(V \mp v)^2;$$

$$\therefore R = \frac{\frac{1}{2} MD (V \mp v)^2}{1 - \frac{1}{2} MD (V \mp v)^2}.$$

*On curvilinear motion in a resisting medium.*

187. If a body move in a resisting medium, impelled by any given forces; its velocity will be that due to the

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* Generally, the medium being given, the resistance varies as $\frac{v^2}{D'\phi}$, where $D'$ is the density of the body, and $\phi$ a direct function of its magnitude. Hence therefore however *small* the body may be, if its density be proportionally *great*, the coefficient of the resistance will remain finite, and, when the velocity of projection is exceeding great (since it can only receive a finite diminution) the expression $P - kv^2$ will, throughout the motion reduce itself to $-kv^2$.

uniform action of their resultant, through one quarter of the chord of curvature which is in the direction of that resultant.

Let $X$ and $Y$ be the forces, resolved in the directions of $x$ and $y$ ($Px$ and $PM$, Fig. 56). Take $PN$ the normal at $P$, and let $PQ$ be the direction of the resultant of $X$ and $Y$. Let the motion of the body be toward $B$;

$$\therefore \frac{d^2x}{dt^2} = -X - R\frac{dx}{ds} \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

$$\frac{d^2y}{dt^2} = -Y - R\frac{dy}{ds} \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2);$$

$$\therefore \frac{dyd^2x - dxd^2y}{dt^2} = Ydx - Xdy;$$

$$\therefore \left(\frac{ds}{dt}\right)^2 = \frac{ds^2}{dyd^2x - dxd^2y}.(Ydx - Xdy)\ldots\ldots(3).$$

Now $\tan QPN = \tan(QPM + MPN)$

$$= \frac{\dfrac{X}{Y} + \dfrac{dy}{dx}}{1 - \dfrac{Xdy}{Ydx}} = \frac{Xdx + Ydy}{Ydx - Xdy};$$

$$\therefore \cos QPN = \frac{Ydx - Xdy}{(X^2 + Y^2)^{\frac{1}{2}}ds};$$

$\therefore \frac{1}{2}$ chord of curvature in the direction $PQ$

$$= \frac{ds^3}{dyd^2x - dxd^2y}.\frac{Ydx - Xdy}{(X^2 + Y^2)^{\frac{1}{2}}.ds} = 2K \text{ Suppose}$$

$$\text{by (3)} \therefore \left(\frac{ds}{dt}\right)^2 = 2K(X^2 + Y^2)^{\frac{1}{2}}.$$

Therefore, the velocity is that due to the uniform action of the force $(X^2 + Y^2)^{\frac{1}{2}}$ through the space $K$.

188. From (equation 1) $\times y$ − (equation 2) $\times x$ we obtain

$$\frac{y d^2 x - x d^2 y}{dt^2} = (Yx - Xy) + R\,\frac{x dy - y dx}{ds};$$

$$\therefore \left(\frac{ds}{dt}\right)^2 = \frac{ds^2}{y d^2 x - x d^2 y}(Yx - Xy) + R\,\frac{x dy - y dx}{y d^2 x - x d^2 y}\,ds \ldots (4).$$

Now $R$ is a function of $\left(\frac{ds}{dt}\right)$, and of the density, also this last is a function of $x$ and $y$. Eliminating $\left(\frac{ds}{dt}\right)$, therefore between the equations (3) and (4) we shall obtain an equation in $x$ and $y$ to the trajectory.

189. If the force $X = 0$; from (equation 1) $\times\, dx$ + (equation 2) $\times\, dy$, (or directly on the principle of the conservation of *vis viva*), we obtain

$$dv^2 = -\,2Ydy - 2Rds \ldots\ldots\ldots\ldots\ldots\ldots (6).$$

Now by (3) $v^2 = \dfrac{Y ds^2 dx}{dy d^2 x - dx d^2 y} = -\,Y\dfrac{ds^2}{d^2 y}$,

considering $dx$ constant;

$$\therefore d\left(\frac{ds^2}{d^2 y}\,Y\right) = 2Ydy + 2Q\,\frac{ds^3}{d^2 y};\quad \text{if } R = Q\left(\frac{ds}{dt}\right)^2.$$

Suppose $Y$ = constant;

$$\therefore 2Ydy - Y\frac{ds^2 d^3 y}{(d^2 y)^2} = 2Ydy + 2Q\,\frac{ds^3}{d^2 y};$$

$$\therefore Y\frac{d^3 y}{(d^2 y)^2} = -\,2Q\,\frac{ds}{d^2 y} \ldots\ldots\ldots\ldots\ldots\ldots (7).$$

190. When the force $P$ tends to a center, the velocity is that due to one quarter of the chord of curvature *through it*.

By (equation 1) × $dx$ + (equation 2) × $dy$, we have

$$d\left(\frac{ds}{dt}\right)^2 = -\ 2Pdr - 2Rds;$$

$$\therefore\ d(PK) = -\ Pdr - Rds,$$

$$\text{also } R = Q\left(\frac{ds}{dt}\right)^2 = 2QPK;$$

$$\therefore\ d(PK) + 2Q\,(PK)\,ds + Pdr = 0.$$

Now if $p$ represent a perpendicular upon the tangent from the center of force,

$$K = \tfrac{1}{2}\frac{pdr}{dp};\quad \therefore\ \frac{dr}{K} = \frac{2dp}{p} = d\text{ h. l. } p^2;$$

$$\therefore\ d\text{ h. l. }(PK) + 2Qds + d\text{ h. l. } p^2 = 0;$$

$$\therefore\ \text{h. l. } \frac{PKp^2}{C} + 2\textstyle\int Qds = 0;$$

$$\therefore\ C\epsilon^{-2\int Qds} = PKp^2 = P\frac{p^3dr}{2dp};$$

$$\therefore\ -\frac{dp^{-2}}{Pdr} = \frac{\epsilon^{2\int Qds}}{C}$$

Now let $u = \frac{1}{r}$; $\therefore\ p^{-2} = \frac{1}{r^2} + \frac{dr^2}{r^4 d\theta^2} = u^2 + \left(\frac{du}{d\theta}\right)^2$;

$$\therefore\ \frac{dp^{-2}}{du} = 2u + \frac{2d^2u}{d\theta^2};\ \therefore\ -\frac{dp^{-2}}{dr} = 2u^3 + 2u^2\frac{d^2u}{d\theta^2} = \frac{P\epsilon^{2\int Qds}}{C};$$

$$\therefore\ \frac{d^2u}{d\theta^2} + u - \frac{P\epsilon^{2\int Qds}}{2Cu^2} = 0.$$

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191. Results deduced on the preceding theory of resistance, are in many cases wholly at variance with experiment.

The following may be assigned as causes of this discrepancy:

192. First, the level of the fluid is not the same at the two opposite surfaces of the body, it is higher than the level of the rest of the stream at the anterior, and lower at the posterior surface.

Let the plane $PQ$ (Fig. 72.) be at rest in a stream of fluid whose motion is in the direction $AM$. Suppose the disturbance produced by the anterior surface to commence at $M$, and that by the posterior surface at $N$. Now considering the line of particles which is in the path of any given particle, taking the plane of $xy$ beneath it, and calling $p$, $z$, $v$ and $p_{,}$, $z_{,}$, $v'$ the pressures, altitudes and velocities corresponding to two given points in it; we have (Art. 132.)

$$p - p_{,} = - Dg(z - z_{,}) \mp \tfrac{1}{2} D (v^2 - v'^2).$$

The sign $\mp$ being taken according as the motion is in the direction in which the co-ordinates are measured and the pressure estimated, or in the opposite direction.

But at the surface of the fluid $p = p_{,} =$ the unit of *atmospheric* pressure;

$$\therefore \; -g(z - z_{,}) \mp \tfrac{1}{2}(v^2 - v'^2) = 0.$$

Now, taking $M$ and $P$ to be the points corresponding to $z$, $v$ and $z_{,}$, $v'$, and using the *negative sign*, we have, since $v' = 0$,

$$-g(z - z_{,}) - \tfrac{1}{2} v^2 = 0.$$

The negative value of $(z - z_{,})$, shows $z_{,}$ to be greater than $z$, or the point $P$ to be above the surface of the rest of the fluid. Also, $v = \sqrt{2g\overline{LP}}$, the fluid is therefore *raised*, at the anterior surface of the plane, to the height due to the velocity of the stream.

Again, at the posterior surface, let the altitudes and velocities corresponding to the points $N$ and $Q$ be $z$, $v_{,}$ and $z_{,}$, $v'$. Taking the *positive sign*, since the pressure is estimated, and the co-ordinates measured from $N$ towards $Q$, or in a direction opposite to the motion, and observing that $v' = 0$, we have

$$-g(z - z_{,}) + \tfrac{1}{2} v^2 = 0.$$

$LQ$ is therefore positive, and the point $Q$ lies *below* the surface of the fluid at the depth $LQ$, due to the velocity at $N$.

193. In addition to the pressure arising from the *inertia* of the fluid; the plane sustains therefore a pressure produced by the *weight* of that which is in contact with the part of it $PQ$.

$$\text{Now } PQ = PL + LQ = \frac{v^2 + v_{,}^2}{2g},$$

$$\therefore \text{ pressure on } PQ = \tfrac{1}{2} g D \overline{PQ}^2 = \frac{D(v^2 + v_{,}^2)^2}{8g}.$$

If $v = v_{,}$ the pressure $= \dfrac{Dv^4}{2g}$. On the whole therefore the resistance $= \dfrac{MDv^2}{2} + \dfrac{mDv^4}{2}$, calling $m$ the width of the plane. If the velocity be considerable, the last term of this expression evidently becomes a principal element of the whole. In this term therefore we find an adequate cause for the difference, *in direct impact*, between the received theory and experiment, which difference is found only where the velocity is great.

All that has been said above, may be extended to the case in which the plane as well as the fluid is in motion, by communicating to both the velocity of the former, in an opposite direction. In this case the resistance

$$= \frac{MD(V \mp v)^2}{2} + \frac{mD(V \mp v)^4}{2g}.$$

194. The second cause of difference between theory and experiment, has reference to the case of oblique impact. It is in this case that the error of the theory is most remarkable. We have supposed the reaction of a surface to take place only in the direction of its normal. There seems no sufficient ground for the application of this hypothesis to the case of fluid resistance. It does not follow that the particles of a solid, however accurately they may be made to arrange themselves in the same plane (which is all that would seem

to be implied in smoothness of surface) should on that account, be effective on the particles of a fluid brought severally in contact with them, otherwise than though they were not so arranged, or even than though they were detached and assumed to themselves the form of another fluid.

## CHAP. VI.

### ON THE MOTION OF ELASTIC FLUIDS.

195. To determine the motion of an elastic fluid into another of different density—both fluids being supposed of infinite extent. In all cases of fluid motion,

$$dp = D\,(dP \mp f ds),$$

$P$ representing the integral $\int(Xdx + Ydy + Zdz)$. Now if $h$ be the height of an homogeneous atmosphere, $p = ghD$;

$$\therefore\ gh\frac{dp}{p} = dP \mp f ds \ldots\ldots\ldots\ldots\ldots\ldots (A).$$

Both fluids being of infinite extent, their density can experience no sensible variation from the *finite* increase or diminution of the *quantity* of either fluid. The *difference* of density which is the cause of motion will therefore remain unaltered, and the motion itself ultimately become uniform, so that *the different particles of the fluid will be impelled by the same accelerating forces as they pass through the same points in space.* Hence precisely as in the case of incompressible fluids it may be shown that $\int f ds = \frac{1}{2}v^2$;

$$\therefore\ gh\ \text{h. l.}\ p = P \mp \tfrac{1}{2}v^2 + C.$$

196. Let the impressed force be that of gravity, and call $p$ and $v$ the common pressure and velocity of the fluids at the aperture.

Therefore in the medium out of which the motion takes place,

$$gh\ \text{h. l.}\ p = gz - \tfrac{1}{2}v^2 + C.$$

In the other medium,

$$gh \text{ h. l. } p = gz' + \tfrac{1}{2}v^2 + C.$$

Suppose $z$ to be measured from a horizontal plane, intersecting the two fluids — $p_1$ and $p_2$ to be the units of pressure at given points in this plane — and the fluid in contact with it to be at rest;

$$\therefore\ gh \text{ h. l. } \frac{p}{p_1} = gz - \tfrac{1}{2}v^2,$$

$$gh \text{ h. l. } \frac{p}{p_2} = gz + \tfrac{1}{2}v^2;$$

$$\therefore\ gh \text{ h. l. } \frac{p_1}{p_2} = v^2.$$

If $D_1$ and $D_2$ be the densities of the media at the plane from which $z$ is measured,

$$\frac{p_1}{p_2} = \frac{D_1}{D_2}; \quad \therefore\ gh \text{ h. l. } \frac{D_1}{D_2} = v^2 \ldots\ldots\ldots\ldots\ldots(B).$$

Let $\delta$ be the density at the aperture;

$$\therefore\ \frac{\delta}{D_1} = \frac{p}{p_1}; \quad \therefore\ gh \text{ h. l. } \frac{\delta}{D_1} = gz - \tfrac{1}{2}v^2$$

$$= gz - \tfrac{1}{2}gh \text{ h. l. } \frac{D_1}{D_2}; \quad \therefore\ gh \text{ h. l. } \left(\frac{\delta}{D_1}\right)\left(\frac{D_1}{D_2}\right)^{\frac{1}{2}} = gz;$$

$$\therefore\ \delta = (D_1 D_2)^{\frac{1}{2}}\, \epsilon^{\frac{z}{h}}.$$

197. If the motion extend but to a short distance above the aperture, so that the plane $xy$ may be so taken, that $z$ may be small when compared with $h$,

$$\delta = (D_1 D_2)^{\frac{1}{2}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(C).$$

Or the density at the common aperture of two elastic fluids, is a mean proportional between the densities of the fluids themselves.

198. We have supposed no variation to take place in the densities of the media from the continual variation in the quantities of fluid they contain. It is clear, that admitting such variation, the motion must no longer be considered uniform. If, however, the aperture be exceeding small, it may be shown as in the case of incompressible fluids, that the error resulting from this cause is inconsiderable and may be neglected. We have therefore generally in the cases of small apertures,

$$v = \sqrt{gh\frac{D_1}{D_2}}, \text{ and } \delta = \sqrt{D_1 D_2}.$$

199. Let there be two vessels, containing air of an uniform density, and let them communicate by means of a common aperture $k$.

Let $M_1$ and $M_2$ be their capacities $D'_1$ and $D'_2$ the *initial* densities of the air, and $D_1$ and $D_2$ the densities after the time $t$.

Now the quantity of air in the first vessel is diminished in the time $t$ by $k\int \delta v dt$;

$$\therefore M_1 D_1 + k\int \delta v dt = M_1 D'_1;$$

$$\therefore k\delta v dt = - M_1 dD_1;$$

$$\therefore t = \frac{M_1}{k\sqrt{gh}} \int \frac{-dD_1}{\sqrt{D_1 D_2 \text{ h. l. } \left(\frac{D_1}{D_2}\right)}},$$

also since the *same quantity* of fluid is continually divided between the two vessels;

$$\therefore M_1 D_1 + M_2 D_2 = M_1 D'_1 + M_2 D'_2.$$

Eliminating $D_2$ between these equations we have an equation in $t_1$ and $D_1$.

200. In the above investigation we have taken into account, that variation in the density of the fluid about the aperture which results from its *motion*.

In the theory commonly adopted, this variation is neglected together with the variation of *pressure* which arises from the *motion* of the fluid in the second vessel.

If we adopt the former part of the common hypothesis, and integrate the equation $A$, considering $D$ as the same for the different values of $f$, we shall obtain,

$$p = g\int D dz \mp \tfrac{1}{2} D v^2.$$

Therefore in the two vessels,

$$p = ghD_1 - \tfrac{1}{2} D_1 v^2,$$

$$\text{and } p = ghD_2 + \tfrac{1}{2} D_2 v^2,$$

and at the aperture,

$$gh(D_1 - D_2) - \tfrac{1}{2}(D_1 + D_2) v^2 = 0;$$

$$\therefore v^2 = 2gh \frac{D_1 - D_2}{D_1 + D_2}.$$

If the fluid pass out of the first vessel into a vacuum, or into a medium very nearly approaching to it, the value of $D_2$ may be neglected, and $v^2 = 2gh$, The velocity is therefore that due to the height of an homogeneous atmosphere. We shall investigate no farther the results which may be deduced on this hypothesis. It is manifestly erroneous.

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201. Suppose a vertical prism (Fig. 64.) containing air to be closed by a piston $PQ$ moveable within it. When the piston is loaded with the weight $W$, let it rest in the position $BC$.

It is required to determine the motion when a weight $\omega$ is in this position added to $W$. Let $AB = a$, $BP = x$. The elasticity of the air in $ABCD$ being represented by $Wg$; that in $APQD$ is represented by $\dfrac{Wga}{a-x}$. The impressed moving force on the piston is therefore,

$$(W + \omega)g - \frac{Wga}{a-x};$$

$$\therefore\ (W+\omega)\,v\,dv = \left\{(W+\omega)\,g - \frac{Wga}{a-x}\right\} dx;$$

$$\therefore\ v^2 = 2g\left\{x + \frac{Wa}{W+\omega}\ \text{h. l.}\ \frac{a-x}{a}\right\}.$$

Taking the integral from 0 to $x$.

In the above we have supposed a vacuum to exist above the piston $PQ$. Let us now take in the consideration of atmospheric pressure. Let $W'g$ represent the weight of the column of air incumbent on $PQ$. Adopting the same notation as before, the elasticity of the air in $ABCD$ is equivalent to the weight $(W+W')\,g$. The elastic force of that in $APQD$ is therefore $\dfrac{(W+W')\,ag}{a-x}$. Also the downward pressure on $PQ$ is $(W+W'+\omega)\,g$, and the mass moved is $(W+\omega)$;

$$\therefore\ (W+\omega)\,v\,dv = \left\{(W+W'+\omega)\,g - \frac{(W+W')\,ag}{a-x}\right\} dx;$$

$$\therefore\ v^2 = 2g\left\{\left(1 + \frac{W'}{W+\omega}\right)x + a\,\frac{W+W'}{W+\omega}\ \text{h. l.}\ \frac{a-x}{a}\right\}.$$

202. To determine the acceleration of a bullet in the barrel of a gun.

Let $P$ (Fig. 65.) be any position of the bullet, and $B$ that in which its motion commenced. Let $Wg$ represent the expansive force of the air in $AB$, $W'g$ the atmospheric pressure, and $\omega g$ the weight of the bullet, $AB=a$, $AP=x$. Then the elastic force of the air in $AP$ is represented by $\dfrac{Wag}{x}$; the impressed moving force by $\dfrac{Wag}{x} - W'g$; and the mass moved by $\omega$;

$$\therefore\ \omega\,v\,dv = \left\{\frac{Wag}{x} - W'g\right\} dx;$$

$$\therefore\ v^2 = 2g\left\{\frac{Wa}{\omega}\ \text{h. l.}\ \frac{x}{a} - \frac{W'}{\omega}\,(x-a)\right\}.$$

In the above we have taken no account of the variation in the elasticity of the fluid produced by the actual motion of its particles, and variable from one point in it to another. Where the expansion is rapid, as in explosion of a cannon, this is evidently a fertile source of error, and accordingly the results we have deduced are found in this case but very imperfectly to agree with experiment.

## CHAP. VII.

### ON THE GENERAL EQUATIONS OF THE MOTION OF FLUIDS.

203. THE fluid mass, the motion of which we are about to consider, may be homogeneous or heterogeneous, incompressible or elastic: all its points are solicited by given forces, such as their mutual attractions, and other attractions directed towards fixed or moveable centers: and the forces which act at any point whose co-ordinates are $x$, $y$, $z$, are reduced to three $X$, $Y$, $Z$, parallel to the three axes $Ox$, $Oy$, $Oz$, of co-ordinates. The quantities $X$, $Y$, $Z$, are simply functions of $x$, $y$, $z$, when the given forces do not change intensity during the motion, and are directed towards fixed centers; but when these forces are directed towards moveable centers, and when they proceed from the mutual attraction of the fluid particles, the values of $X$, $Y$, $Z$, will involve the time in their expressions. In general then $X$, $Y$, $Z$, will be functions of $x$, $y$, $z$, and $t$.

Let us also resolve into components parallel to the axes $Ox$, $Oy$, $Oz$, the velocity which corresponds to the co-ordinates $x$, $y$, $z$; and let $u$, $v$, $w$, be the respective components. These will be unknown functions of $x$, $y$, $z$, $t$; they will depend on the co-ordinates $x$, $y$, $z$, because for the same value of $t$, the velocity varies from one particle to another in magnitude and direction; they will depend also on the time $t$, because for the same values of $x$, $y$, $z$, the velocity changes from one instant to another. If we wish to compare with one another, the velocities of the same particle, at two suc-

cessive instants, we must suppose the variable $t$ to become $t+dt$, and at the same time the co-ordinates $x, y, z$ of this particle, to become $x+udt$, $y+vdt$, $z+wdt$: for in virtue of the velocities $u, v, w$, the same particle which answered to the co-ordinates $x, y, z$, at the end of the time $t$, will answer to the co-ordinates $x+udt$, $y+vdt$, $z+wdt$, at the end of the time $t+dt$. It follows then, that to have the variation of $u, v, w$, the component velocities of a certain particle, we must differentiate with respect to $t$, and with respect to $x, y, z$, taking $udt$, $vdt$, $wdt$, for the increments of these latter variables. In this manner we shall have,

$$du = \frac{du}{dt}dt + \frac{du}{dx}udt + \frac{du}{dy}vdt + \frac{du}{dz}wdt,$$

$$dv = \frac{dv}{dt}dt + \frac{dv}{dx}udt + \frac{dv}{dy}vdt + \frac{dv}{dz}wdt,$$

$$dw = \frac{dw}{dt}dt + \frac{dw}{dx}udt + \frac{dw}{dy}vdt + \frac{dw}{dz}wdt.$$

Let us divide the fluid mass into rectangular parallelopipeds, infinitely small, the sides of which are parallel to $Ox$, $Oy$, $Oz$. The volume of the element which answers to the co-ordinates $x, y, z$, will have for expression the product $dxdydz$. We may regard the density constant throughout the extent of this volume, and designating it by $\rho$, the mass of the element will be $\rho dxdydz$. Let us also represent by $p$, the pressure referred to the unit of surface, which the surrounding fluid exercises on the different faces of this parallelopiped, and which is the same on all the faces, according to the fundamental property of fluids. The two quantities $\rho$ and $p$ are, as well as the velocities $u, v, w$, unknown functions of $x, y, z$ and $t$. The five quantities $u, v, w, \rho$ and $p$, are the unknown quantities of the problem which occupies us: when they shall be determined in functions of $x, y, z$, and $t$, the state of the fluid mass will be known at each instant, because we shall then know the velocity, its direction, the density of the fluid, and the pressure it exerts, at whatever point we choose to fix upon, whether

at the surface or in the interior of the mass. Let us seek then the equations on which the values of these five quantities depend.

204. Three of these equations are immediately furnished by D'Alembert's principle. The velocities lost during the instant $dt$ by the particle submitted to the action of the forces $X$, $Y$, $Z$, are $Xdt-du$, $Ydt-dv$, $Zdt-dw$; for $du$, $dv$, $dw$, express the increments of velocity which really take place during this instant, and $Xdt$, $Ydt$, $Zdt$, are those which would be produced by the forces $X$, $Y$, $Z$, if the particle were free. Dividing these velocities by $dt$, in order to have the measure of the accelerative forces capable of producing them; designating these forces by $X'$, $Y'$, $Z'$; and putting for $du$, $dv$, $dw$, their values above; we find,

$$X-\frac{du}{dt}-\frac{du}{dx}u-\frac{du}{dy}v-\frac{du}{dz}w=X',$$

$$Y-\frac{dv}{dt}-\frac{dv}{dx}u-\frac{dv}{dy}v-\frac{dv}{dz}w=Y',$$

$$Z-\frac{dw}{dt}-\frac{dw}{dx}u-\frac{dw}{dy}v-\frac{dw}{dz}w=Z'.$$

But, according to D'Alembert's principle, equilibrium will have place in the fluid mass, if all the particles be solicited by forces capable of impressing on them the velocities lost or gained at each instant: the general equation of the equilibrium of fluids found in Art. 94, ought then to be satisfied, when we take $X'$, $Y'$, $Z'$ for the accelerative forces parallel to the axes of co-ordinates. Thus

$$(dp)=\rho\,(X'dx+Y'dy+Z'dz).$$

And if $\frac{dp}{dx}$, $\frac{dp}{dy}$, $\frac{dp}{dz}$, be the partial differential coefficients of $p$ with respect to $x$, $y$, $z$, respectively,

$$\frac{dp}{dx}=\rho X',\ \frac{dp}{dy}=\rho Y',\ \frac{dp}{dz}=\rho Z'.$$

Hence, putting for $X'$, $Y'$, $Z'$, their values, and dividing by $\rho$,

$$\left.\begin{aligned}\frac{1}{\rho}\cdot\frac{dp}{dx} &= X - \frac{du}{dt} - \frac{du}{dx}u - \frac{du}{dy}v - \frac{du}{dz}w\\ \frac{1}{\rho}\cdot\frac{dp}{dy} &= Y - \frac{dv}{dt} - \frac{dv}{dx}u - \frac{dv}{dy}v - \frac{dv}{dz}w\\ \frac{1}{\rho}\cdot\frac{dp}{dz} &= Z - \frac{dw}{dt} - \frac{dw}{dx}u - \frac{dw}{dy}v - \frac{dw}{dz}w\end{aligned}\right\}\ldots\ldots(a).$$

205. Each of the elements into which the fluid mass was supposed to be divided, will change form during the instant $dt$, and it will also change volume if the fluid be compressible: but as its mass ought always to remain the same, it follows that if we seek what its volume and its density become at the end of the time $t+dt$, their product ought to be the same as at the end of the time $t$. Making then equal to zero the variation of this product, there will result a new equation of the motion.

To form this equation, let us consider the rectangular parallelopiped, of which the volume was expressed by $dxdydz$, at the end of the $t$, and let us see the shape that this portion of the fluid will take at the end of the time $t + dt$. There will, of course, be eight solid angles of the parallelopiped, and as many angular summits. I call that summit $m$, which is nearest $O$, the origin of co-ordinates, and consider its co-ordinates to be exactly $x$, $y$, $z$. Let that summit, which lies from $m$ in the direction of $z$, be called $n$. At the end of the instant $dt$, the co-ordinates of $m$ become,

$$x + udt,\ \ y + vdt,\ \ z + wdt:$$

at the end of the same instant the co-ordinates of $n$ become,

$$x + u'dt,\ \ y + v'dt,\ \ z + dz + w'dt;$$

$u'$, $v'$, $w'$, being the values of $u$, $v$, $w$, at the summit $n$. But

$$u' = u + \frac{du}{dz}dz,\ \ v' = v + \frac{dv}{dz}dz,\ \ w' = w + \frac{dw}{dz}dz,$$

because the alteration of $u$, $v$, $w$, from $m$ to $n$ is, at a given instant, relative only to $z$. Hence the co-ordinates of $n$ at the end of $dt$ become

$$x + u\,dt + \frac{du}{dz}\,dz\,dt, \quad y + v\,dt + \frac{dv}{dz}\,dz\,dt,$$

$$z + dz + w\,dt + \frac{dw}{dz}\,dz\,dt;$$

and the distance between $m$ and $n$ at the same time, is,

$$\sqrt{\left(\frac{du}{dz}\right)^2 dz^2 dt^2 + \left(\frac{dv}{dz}\right)^2 dz^2 dt^2 + \left(dz + \frac{dw}{dz}\,dz\,dt\right)^2}.$$

Extracting the square root, and neglecting the infinitely small terms of the third order, we have for this distance

$$dz + \frac{dw}{dz}\,dz\,dt.$$

Now let us consider the two summits which lie in the same diagonal plane of the parallelopiped as $m$ and $n$, and let us call them $m'$ and $n'$, $m'$ being that which is nearest the plane $xOy$. The parallelopiped being in its original position, the co-ordinates of $m'$ are $x + dx$, $y + dy$, $z$; those of $n'$, $x + dx$, $y + dy$, $z + dz$. Hence, to obtain the distance between $m'$ and $n'$ at the end of $dt$, $x + dx$ and $y + dy$ must be substituted for $x$ and $y$ in the preceding value of the distance between $m$ and $n$. By this substitution $\frac{dw}{dz}$ becomes $\frac{dw}{dz} + \frac{d^2w}{dz\,dx}\,dx + \frac{d^2w}{dz\,dy}\,dy$. Hence the distance sought is,

$$dz + \frac{dw}{dz}\,dz\,dt + \frac{d^2w}{dz\,dx}\,dx\,dz\,dt + \frac{d^2w}{dz\,dy}\,dy\,dz\,dt.$$

Neglecting then the two last terms, which are of the third order, the distance at the end of $dt$, between $m'$ and $n'$, is the same as that between $m$ and $n$. Thus the two sides connecting $m$, $n$, and $m'$, $n'$, which were equal at the com-

mencement of the increment of time $dt$, continue to be equal at the end of that instant, excepting so far as regards infinitely small quantities of an order which may be neglected. By similar reasoning it would be found that the two sides parallel and equal to these, continue to be equal to them during the same increment of time. In like manner it may be shewn, that in each of the other two sets of parallel sides, the equality of the sides is preserved during a very small motion. The value of each side in one of these sets at the end of $dt$, is found by changing $z$ into $y$ and $w$ into $v$, in the value of the distance between the summits $m$, $n$, just obtained; so that we shall have

$$dy + \frac{dv}{dy} dy dt:$$

the value of each side in the other is similarly found by changing $z$ into $x$ and $w$ into $u$; we shall have

$$dx + \frac{du}{dx} dx dt.$$

We obtain the volume of the element by multiplying the face, which was originally parallel to the plane $xOy$, and which passes through $m$, by the altitude of the summit $n$ above this face. The area of the face is the product of the two sides meeting in the summit $m$, multiplied by the sine of the angle contained between them; the altitude of the summit is equal to the side which joins $m$ and $n$ multiplied by the sine of the angle it makes with the plane of the face. Hence the volume of the parallelopiped will be the product of the three sides which meet in $m$, multiplied by the product of the sines of the two angles just mentioned. But as these angles were right angles in the original parallelopiped, neither of them can differ from $90^0$ but by a quantity indefinitely small: their sines will consequently differ from unity by quantities indefinitely small of the second order, which may be neglected. The volume sought will therefore be,

$$\left(dx + \frac{du}{dx} dx dt\right) \left(dy + \frac{dv}{dy} dy dt\right) \left(dz + \frac{dw}{dz} dz dt\right).$$

Performing the multiplication and neglecting terms of the fifth order, there will arise

$$dxdydz\left(1+\frac{du}{dx}dt+\frac{dv}{dy}dt+\frac{dw}{dz}dt\right);$$

which is the volume at the end of the time $t+dt$, of the element which was $dxdydz$ at the end of $t$. As $\rho$ is a function of $x$, $y$, $z$, and $t$, it follows that when $t$ becomes $t+dt$, and at the same time $x$, $y$, $z$, are changed into $x+udt$, $y+vdt$, $z+wdt$, the density becomes,

$$\rho+\frac{d\rho}{dt}dt+\frac{d\rho}{dx}udt+\frac{d\rho}{dy}vdt+\frac{d\rho}{dz}wdt.$$

As the mass of the same element remains constant, if this density be multiplied by the corresponding volume, and the product be diminished by $\rho dxdydz$, the remainder ought to be equal to zero. If in performing the operation, terms involving $dt^2$ be neglected, and the factors $dxdydz$ and $dt$ common to all the terms be removed, we shall find,

$$\frac{d\rho}{dt}+\frac{d\rho}{dx}u+\frac{d\rho}{dy}v+\frac{d\rho}{dz}w+\rho\frac{du}{dx}+\rho\frac{du}{dy}+\rho\frac{du}{dz}=0.$$

When the fluid is incompressible, and either homogeneous or heterogeneous, the variation of the density as well as the mass of the same element is nothing: so that,

$$\left.\begin{array}{l}\dfrac{d\rho}{dt}+\dfrac{d\rho}{dx}u+\dfrac{d\rho}{dy}v+\dfrac{d\rho}{dz}w=0\\[2ex]\dfrac{du}{dx}+\dfrac{du}{dy}+\dfrac{du}{dz}=0\end{array}\right\}(c).$$

When the fluid is elastic $p$ is a function of $\rho$. In both cases, then, we have two equations, which joined to the three equations $(a)$ serve to determine $p$, $\rho$, $u$, $v$, $w$, in terms of $x$, $y$, $z$ and $t$.

206. When the fluid is homogeneous and incompressible, the first of equations $(c)$ becomes identical; and we have simply the second equation and the three equations $(a)$

to determine $p$, $u$, $v$, $w$. To this case let us now proceed to direct our attention more particularly. And first, it is to be observed that the differential equations of the motion will be found to be susceptible of great simplification whenever $udx + vdy + wdz$ is a complete differential of a function of $x$, $y$, $z$. We shall assume this to be the case at present, and afterwards consider to what circumstance of the motion this analytical fact has reference.

Let $udx + vdy + wdz = d\phi$, $\phi$ being a function of $x, y, z$, which may besides contain $t$, but which is not differentiated with respect to this variable. Then,

$$u = \frac{d\phi}{dx}, \quad v = \frac{d\phi}{dy}, \quad w = \frac{d\phi}{dz}.$$

Hence the second of equations ($c$) becomes

$$\frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2} + \frac{d^2\phi}{dz^2} = 0 \ldots\ldots\ldots\ldots\ldots (d).$$

Again, as $udx + vdy + wdz = d\phi$, $\frac{du}{dt}dx + \frac{dv}{dt}dy + \frac{dw}{dt}dz$ is equal to

$$\frac{d^2\phi}{dxdt}dx + \frac{d^2\phi}{dydt}dy + \frac{d^2\phi}{dzdt}dz,$$

which is

$$\frac{d \,.\, \frac{d\phi}{dt}}{dx}dx + \frac{d \,.\, \frac{d\phi}{dt}}{dy}dy + \frac{d \,.\, \frac{d\phi}{dt}}{dz}dz, \text{ or } d \,.\, \frac{d\phi}{dt},$$

the differentiation of $\frac{d\phi}{dt}$ being with respect to $x$, $y$, $z$, whilst $t$ is constant. Thus,

$$\frac{du}{dt}dx + \frac{dv}{dt}dy + \frac{dw}{dt}dz = d \,.\, \frac{d\phi}{dt}$$

$$\text{so} \quad \frac{du}{dx}dx + \frac{dv}{dx}dy + \frac{dw}{dx}dz = d \,.\, \frac{d\phi}{dx}$$

$$\frac{du}{dy}dy + \frac{dv}{dy}dy + \frac{dw}{dy}dz = d\,.\,\frac{d\phi}{dy}$$

$$\frac{du}{dz}dz + \frac{dv}{dz}dz + \frac{dw}{dz}dz = d\,.\,\frac{d\phi}{dz}$$

Hence, if we add equations ($a$), after having multiplied the first by $dx$, the second by $dy$, the third by $dz$, we shall have,

$$\frac{1}{\rho}.dp = Xdx + Ydy + Zdz - d.\frac{d\phi}{dt} - ud.\frac{d\phi}{dx} - v.d.\frac{d\phi}{dy} - w.d.\frac{d\phi}{dz};$$

and putting $\frac{d\phi}{dx}$, $\frac{d\phi}{dy}$, $\frac{d\phi}{dz}$, in the place of $u$, $v$, $w$, this equation may be thus written;

$$\frac{dp}{\rho} = Xdx + Ydy + Zdz - d.\frac{d\phi}{dt} - \tfrac{1}{2}.d.\left(\frac{d\phi^2}{dx^2} + \frac{d\phi^2}{dy^2} + \frac{d\phi^2}{dz^2}\right).$$

The differentials of $p$, $\frac{d\phi}{dt}$, and $\frac{d\phi^2}{dx^2} + \frac{d\phi^2}{dy^2} + \frac{d\phi^2}{dz^2}$, which are indicated in this equation, ought to be taken with respect to $x$, $y$, $z$, without making $t$ vary.

It is permitted to suppose the formula $Xdx + Ydy + Zdz$ a complete differential with respect to $x$, $y$, $z$, of a function of these variables and of $t$, since it is so in the case of attractive forces directed to fixed or moveable centers; and these comprehend all the forces in nature which can act upon the particles of the fluid mass. Let therefore

$$Xdx + Ydy + Zdz = dV.$$

We shall have by integrating all the terms of the equation above,

$$\frac{p}{\rho} = V - \frac{d\phi}{dt} - \tfrac{1}{2}\left(\frac{d\phi^2}{dx^2} + \frac{d\phi^2}{dy^2} + \frac{d\phi^2}{dz^2}\right)\ldots\ldots\ldots(e).$$

As the integration is relative to $x$, $y$, $z$, an arbitrary function of $t$ should be added: but this may be supposed to be included in $\frac{d\phi}{dt}$.

207. An exact integral of $\frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2} + \frac{d^2\phi}{dz^2} = 0$, may be obtained by supposing $\phi$ to be a function of $x^2 + y^2 + z^2$ and $t$.

Let $x^2 + y^2 + z^2 = r^2$, so that $\phi$ becomes a function of $r$ and $t$. Then

$$\frac{d\phi}{dx} = \frac{d\phi}{dr} \cdot \frac{dr}{dx} = \frac{d\phi}{dr} \cdot \frac{x}{r}: \quad \frac{d^2\phi}{dx^2} = \frac{d^2\phi}{dr^2} \cdot \frac{x^2}{r^2} + \frac{d\phi}{dr}\left(\frac{1}{r} - \frac{x^2}{r^3}\right).$$

Hence

$$\frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2} + \frac{d^2\phi}{dz^2} = \frac{d^2\phi}{dr^2} \cdot \frac{x^2 + y^2 + z^2}{r^2} + \frac{d\phi}{dr}\left(\frac{3}{r} - \frac{x^2 + y^2 + z^2}{r^3}\right)$$

$$= \frac{d^2\phi}{dr^2} + \frac{2d\phi}{dr} = \frac{d^2 . r\phi}{rdr^2} = 0.$$

Integrating, $\frac{d . r\phi}{dr} = f(t)$; $r\phi = f(t)r + F(t)$;

$$\therefore \phi = f(t) + \frac{F(t)}{\sqrt{x^2 + y^2 + z^2}}.$$

This is the proper integral of the equation ($d$); for the supposition that $\phi$ is a function of $r$ and $t$, is an artifice for effecting the integration, the legitimacy of which is proved by the value of $\phi$ thereby obtained. As this integral has been found whilst the origin of co-ordinates is arbitrary, and independently of any supposition about the manner of disturbing the fluid, it must receive a general interpretation, and be understood as relating to the mode of action of the parts of the fluid on each other. The velocity

$$= \sqrt{\frac{d\phi^2}{dx^2} + \frac{d\phi^2}{dy^2} + \frac{d\phi^2}{dz^2}} = \frac{d\phi}{dr} = \frac{-F(t)}{r^2}.$$

By this result we are taught that the motions of the particles in every very small portion of the fluid are directed to or from a point, and their velocities vary inversely as the squares of the distances from this point.

This law of the action of the parts of the fluid on each other, may be verified by conceiving a spherical mass of the fluid to be inclosed in an expansible envelope, and a small solid sphere to be placed at the center of the mass concentric with it. By the insertion of the solid ball the fluid particles will be made to move from their original places through spaces, which vary inversely as the square of the distances from the center.

It is to be observed, that the above law of the motion has been arrived at by supposing, that $u\,dx + v\,dy + w\,dz$ is a complete differential of a function of $x$, $y$, $z$; and such will be the case when the motion is directed to or from fixed or moveable centers, for the same reason that $X dx + Y dy + Z dz$ is a complete differential of a function of $x$, $y$, $z$, when the forces are directed to fixed or moveable centers. The motion resulting from the action of the parts of the fluid on each other, is found to be of this very kind; hence the legitimacy of the supposition is established. The equations ($d$) and ($e$) are however inapplicable, when the particles do not change their relative positions by their mutual action; that is, when the fluid mass moves as a solid would. In this case $\phi$ has no longer existence, and the pressure is determined by

$$\frac{p}{\rho} = \int (X dx + Y dy + Z dz) + C,$$

$X$, $Y$, $Z$ including the forces resulting from rotation.

208. In order farther to illustrate the nature of the integral obtained above, I will consider the case in which the motion is in space of two dimensions.

Let the plane of motion be that of $xy$. Then we shall have simply

$$\frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2} = 0.$$

Assuming, as before, that $y$ is a function of $t$ and $x^2 + y^2$ or $r^2$,

$$\frac{d\phi}{dx} = \frac{d\phi}{dr} \cdot \frac{dr}{dx} = \frac{d\phi}{dr} \cdot \frac{x}{r};$$

$$\frac{d^2\phi}{dx^2} = \frac{d^2\phi}{dr^2} \cdot \frac{x^2}{r^2} + \frac{d\phi}{dr} \cdot \left(\frac{1}{r} - \frac{x^2}{r^3}\right).$$

Hence $\dfrac{d^2\phi}{dx^2} + \dfrac{d^2\phi}{dy^2} = \dfrac{d^2\phi}{dr^2} + \dfrac{d\phi}{rdr} = 0.$

But $\dfrac{d \cdot \frac{rd\phi}{dr}}{dr} = \dfrac{d\phi}{dr} + \dfrac{rd^2\phi}{dr^2}$. Hence $\dfrac{d \cdot \frac{rd\phi}{dr}}{rdr} = 0.$

Integrating, $\dfrac{rd\phi}{dr} = f(t)$, $\phi = f(t) \log r + F(t)$.

It thus appears, that the velocity $\dfrac{d\phi}{dr}$ is $\dfrac{f(t)}{r}$. The meaning of this result may be illustrated as before, by conceiving fluid to be contained in a cylinder capable of expansion in the direction of the radii, and a very slender cylinder of solid matter to be placed with its axis coincident with the other. The fluid particles, by the insertion of the solid cylinder, will be moved from their original positions through spaces which vary inversely as the distances from the axis.

As the integral of $\dfrac{d^2\phi}{dx^2} = -\dfrac{d^2\phi}{dy^2}$ is also (see *Lacroix*, Art. 319.)

$$\phi = F(x - \sqrt{-1}\,y) + f(x + \sqrt{-1}\,y),$$

it is important to shew, that when the origin and direction of co-ordinates are indeterminate, this amounts to the one already found.

$$\frac{d\phi}{dx} = F'(x - \sqrt{-1}\,y) + f'(x + \sqrt{-1}\,y)$$
$$= A - B\sqrt{-1} + A' + B'\sqrt{-1}$$

$$\frac{d\phi}{dy} = -\sqrt{-1}\,F'(x - \sqrt{-1}\,y) + \sqrt{-1}\,f'(x + \sqrt{-1}\,y)$$
$$= -A\sqrt{-1} - B + A'\sqrt{-1} - B'.$$

Hence $\frac{d\phi}{dx}$ and $\frac{d\phi}{dy}$ cannot both be possible, unless $A = A'$ and $B = B'$; that is, unless $F'$ and $f'$ be the same functions. As the direction of axes is arbitrary, let $y = 0$; then

$$\frac{d\phi}{dx} = 2F'(x), \text{ and } \frac{d\phi}{dy} = 0.$$

This proves that the velocity is directed to or from the origin of co-ordinates, and is equal to twice a function of the distance of the same form as $F'$. Hence,

$$\frac{d\phi}{dy} = \sqrt{-1}\{F'(x + y\sqrt{-1}) - F'(x - y\sqrt{-1})\} = \frac{2y}{r}F'(r).$$

Let $x + y\sqrt{-1} = m$, $x - y\sqrt{-1} = n$; so that

$$2y = (n - m)\sqrt{-1} \text{ and } r^2 = mn.$$

Then, $F'(m) - F'(n) = \frac{n - m}{\sqrt{mn}}F'(\sqrt{mn})$

$$= \sqrt{\frac{n}{m}}F'(\sqrt{mn}) - \sqrt{\frac{m}{n}}F'(\sqrt{mn}).$$

As this equation is identical, $F'(m)$ is the same as

$$\sqrt{\frac{n}{m}} \times F'(\sqrt{mn}).$$

Hence $F'(\sqrt{mn})$ must $= \frac{C}{\sqrt{mn}}$: and the velocity

$$= 2F'(r) = \frac{2C}{r},$$

$C$ being an arbitrary function of the time. This is the same result as was obtained before in a different manner.

209. The integral in Art. 207, which is general in regard to the peculiar character of the motion of the fluid, having been thus obtained, we may proceed to apply it to particular cases. For this purpose, an origin of co-ordinates and direction of the axes must be fixed upon.

Let $a$, $\beta$, $\gamma$ be the co-ordinates of the point from which or towards which the motion at the point whose co-ordinates are $x$, $y$, $z$ tends. Then

$$\phi = f(t) + \frac{F(t)}{\sqrt{(x-a)^2 + (y-\beta)^2 + (z-\gamma)^2}}.$$

The equation $\frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2} + \frac{d^2\phi}{dz^2} = 0$, will be satisfied by this value of $\phi$, for $a$, $\beta$, $\gamma$ are in general functions of $x$, $y$, $z$, and the time, but may be considered constant for an instant, while $x$, $y$, $z$ vary a little, for the same reason that the center of curvature of a curve is a constant point, whilst the co-ordinates vary a little. The chief difficulty in the complete solution of a proposed problem, is the ascertaining from the data the values of $a$, $\beta$, $\gamma$. But some circumstances of the motion may be arrived at without the knowledge of these functions, as I am about to shew in the following instance.

If any part of the motion be known to be rectilinear, we may consider this part by itself, whatever be the rest of the motion; for the arbitrary functions which occur in the integral, teach us that there is no necessary connexion between the motion at one point and the motion at another, excepting such as we choose to impose. Suppose fluid to issue from a small circular orifice at the bottom of a vessel, the form of which is generated by the revolution of any irregular line about an axis. The axis is supposed to pass through the center of the orifice. It is plain that along the axis the motion will be rectilinear, as there is no reason why there should be deviation in one direction rather than another. Let this line be the axis of $x$. Then $v=0$, $w=0$, $z=0$, $y=0$, $\beta=0$, $\gamma=0$;

$$\phi = f(t) + \frac{F(t)}{x-a}, \text{ and } V = gx.$$

For simplicity, I will take the case in which the fluid is retained at a constant elevation, and the motion has attained its ultimate state, so that the velocity is the same of all particles

passing through the same point. Then because the velocity $= \frac{d\phi}{dx} = -\frac{F(t)}{(x-a)^2}$, in the case supposed both $F(t)$ and $a$ are independent of $t$, and $\frac{d\phi}{dt} = f'(t)$. The equation ($e$) consequently becomes, supposing $\rho = 1$,

$$p = gx - \frac{u^2}{2} - f'(t).$$

Also it is permitted to consider independently of the rest of the motion, that which takes place along any irregular line, always drawn in the direction of the motion of the particles through which it passes. In this instance, when the fluid has arrived at its ultimate state of motion, $F(t)$, $a$, $\beta$, $\gamma$, are all independent of $t$. Hence as before $\frac{d\phi}{dt} = f'(t)$, and if $q =$ the velocity,

$$p = gx - \frac{q^2}{2} - f'(t).$$

This agrees with Chap. II. Art. 137.

210. With respect to compressible fluids, the equations in the general case are of too complicated a nature to be treated of here. Happily however, the most interesting case, that in which the motions are small, and no extraneous force acts, presents the fewest mathematical difficulties. In this case we may omit in the equation (Art. 205.) the terms involving $u$, $v$, $w$, as factors, because $\frac{d\rho}{dx}$, $\frac{d\rho}{dy}$, $\frac{d\rho}{dz}$, must also be small.

$$\text{Hence } \frac{d\rho}{\rho dt} + \frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0;$$

or supposing $d\phi = u\,dx + v\,dy + w\,dz$,

$$\frac{d.\text{h.l.}\,\rho}{dt} + \frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2} + \frac{d^2\phi}{dz^2} = 0.$$

$$\text{Let } p = a^2\rho. \quad \text{Then} \int\frac{dp}{\rho} = a^2\int\frac{d\rho}{\rho}.$$

Hence equation ($d$) becomes, neglecting the square of the velocity,

$$a^2\int\frac{d\rho}{\rho} = -\frac{d\phi}{dt}.$$

$$\text{Hence } \frac{a^2 d\,.\,\text{h. l.}\,\rho}{dt} = -\frac{d^2\phi}{dt^2},$$

$$\text{so that } \frac{d^2\phi}{dt^2} = a^2\left(\frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2} + \frac{d^2\phi}{dz^2}\right).$$

Here again, as in the equation for incompressible fluids, an exact integral may be obtained by supposing $\phi$ to be a function of $r$ and $t$; and the integral, as before, has reference to the circumstance that the motion of every very small portion of the fluid is directed to or from a fixed or moveable center. We shall arrive at the equation

$$\frac{d^2\,.\,r\phi}{dt^2} = a^2\,.\,\frac{d^2\,.\,r\phi}{dr^2};$$

the integral of which is,

$$r\phi = F(r - at) + f(r + at), \text{ (\textit{Lacroix}, Art. 319.)}$$

The function $F$ applies to a motion of propagation from the center, the function $f$ to a motion of propagation towards the center, as will be understood from the discussion of the analogous equations for motion in space of one dimension, which is given in the Appendix.

# CHAP. VIII.

## HYDROSTATICAL INSTRUMENTS.

### BRAMAH'S HYDROSTATIC PRESS.

211. $L$ and $H$ (Fig. 35.) are vertical cylindrical cavities in a solid mass $EF$ of metal or other strong material. The diameter of $H$ is considerably less than that of $L$, and they communicate through a pipe $MN$. $LA$ is a strong piston or solid cylinder of iron fitting closely to the surface of the cylinder $L$, moveable in it, and terminating in an extended surface at $B$, where the pressure of the instrument is applied. $CH$ is a piston similarly applied to the other cavity $H$, and moveable by means of a lever $DO$, whose fulcrum is at $O$. At $H$ is a valve closing downwards, and beyond it the cylinder is continued to a reservoir $G$. The channel $MN$ contains a valve closing in the direction $MN$. The lever $DO$ being raised, the valve $H$ opens, and water is made to ascend, as in the common pump, from the reservoir $G$ into the cavity $H$. The lever being then *pressed down*, the valve closes, and the water is forced through the channel $NM$ beneath the piston $L$. The whole of the fluid having been expelled from $H$, the piston is drawn up, and the operation repeated.

The pressure thus produced on the bottom of the piston at $L$, and effective at $B$, is to the force impressed at $H$, as the section of the piston $HC$ to that of the piston $LA$.

212. Let $R$ and $r$ be the radii of the pistons $LA$ and $HC$, $L$ the length of the lever $OD$, $l$ the distance of the piston rod from $O$, and $P$ the power applied at $D$;

Therefore the pressure produced by $P$ at $H = \frac{L}{l} P$,

$$\text{pressure at } B = \frac{R^2}{r^2} \,.\, (\text{pressure at } H);$$

$$\text{therefore pressure at } B = \frac{LR^2}{lr^2} \,.\, P.$$

This press appears to present the simplest and most effective * of all mechanical contrivances for increasing human power.

The limit to its practical *application* is found in the extreme slowness of the ascent of the piston $B$, when the ratio of the radii of the cylinders is considerable.

### THE HYDROMETER.

213. The Hydrometer is used for determining the specific gravities of fluids. In its simplest form it consists of a hollow sphere $C$, (Fig. 57.) one of whose diameters is prolonged in the stem $BA$, and to the other extremity of this diameter there is attached a second sphere $D$, so loaded with shot or quicksilver as to keep the instrument in a vertical position, and allow of its floating in the fluids to be examined.

Now if the instrument be made to float in *distilled water* it will sink to a point $A$, such that the fluid it displaces may be equal to its weight. Let the distance $AB$ be divided into any number of equal parts, and the divisions marked on the stem and continued upwards from $A$. Now suppose the instrument to float in a fluid, in which it sinks to the point $P$ in the stem, distant $x$ divisions from $B$. Calling $M$ the mass of the balls $A$ and $B$, $k$ the volume intercepted between each two divisions of the stem, $D$ the density of water, $D'$ that of the fluid, $s$ its specific gravity, and $a$ the number

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* If the cylinder $H$ be $\frac{1}{4}$ of an inch, and $L$ a yard in diameter, a force of 41472 tons may be produced at $B$ by a pressure of two tons at $H$.

of divisions in the portion of the stem $AB$; the weight of the mass of fluid displaced, when the instrument is immersed in water will be represented by $(ak+M)\,Dg$, and when it is immersed in the other fluid, by $(xk+M)\,D'g$. Now each of these quantities is equal to the weight of the instrument;

$$\therefore\ (xk+M)\,D' = (ak+M)\,D;$$

$$\therefore\ s = \frac{D'}{D} = \frac{ak+M}{xk+M} = \frac{a+\dfrac{M}{k}}{x+\dfrac{M}{k}}.$$

214. The most convenient method of adjusting and dividing the instrument is, commencing the division from any point in the stem, which circumstances may point out as most convenient, to continue it upwards at equal intervals, and then to determine the constants $a$ and $\frac{M}{k}$, by immersing the instrument successively in two different fluids, whose specific gravities $s'$ and $s''$ are known. If $x'$ and $x''$ be observed values of $x$ measured from the point where the division commences, and $M$ the whole mass of the instrument below that point, we have

$$s' = \frac{a+\dfrac{M}{k}}{x'+\dfrac{M}{k}}, \qquad s'' = \frac{a+\dfrac{M}{k}}{x''+\dfrac{M}{k}}.$$

From which equations $a$ and $\frac{M}{k}$ may be determined, and the specific gravity corresponding to each division ($x$) marked on the stem.

215. Resuming the equation $s = \dfrac{a+\dfrac{M}{k}}{x+\dfrac{M}{k}}$, we obtain

$$x+\frac{M}{k} = \frac{a+\dfrac{M}{k}}{s}.$$

Differentiating with regard to $x$ and $s$,

$$dx = - \,. \frac{a + \frac{M}{k}}{s^2}\, ds.$$

Whence it appears that for any given small variation $ds$ in the specific gravity of the fluid in which the instrument is immersed, the corresponding variation in the depth of immersion varies as $\frac{a + \frac{M}{k}}{s^2}$, which quantity may be considered a measure of the *susceptibility* of the instrument. It increases as the weight of the whole and the length of the portion of the stem below the first point of division increase, and as the specific gravity of the fluid and the radius of the stem diminish.

216. *The Areometer of Mr. de Parcieux* is in fact an *Hydrometer* of an extraordinary susceptibility produced by the extreme *slenderness* of its stem. The fluid whose specific gravity is to be determined by this instrument is placed in a cylindrical glass vessel, along its side a scale of equal parts is graduated, and the depth of the immersion is found by observing the division of this scale opposite to the extremity of the stem*.

217. The principal obstacle to the use of the simple Hydrometer, is the inconvenience and difficulty of calculating and marking against the different divisions of the stem of each instrument, a different scale of specific gravity, and constructing the stem of that perfectly uniform thickness, which is necessary to the accuracy of the observations.

218. To obviate these difficulties *Fahrenheit* conceived the idea of sinking the *Hydrometer* always to the *same depth*

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* Such is the extreme delicacy of this instrument that the variation of density produced by the falling of the Sun's rays on water of the common temperature, will instantly cause it to sink some inches in the fluid.

by means of a weight to be placed in a cup at the end of the stem.

Calling $W$ the weight of this instrument $\omega'$ and $\omega$ the weights respectively necessary to sink it to the *same* given depth in water, and in the fluid whose specific gravity ($s$) is required to be determined, $M$ the constant bulk of the portion immersed, and $D'$ and $D$ the densities of water and the fluid, we have

$$gDM = W + \omega,$$

$$gD'M = W + \omega';$$

$$\therefore s = \frac{D}{D'} = \frac{W + \omega}{W + \omega'}.$$

219. Differentiating, we get

$$d\omega = (W + \omega')\, ds = gD'M\, ds.$$

Hence it appears that for a given small variation $ds$ of the specific gravity of the fluid, the variation of the weight $\omega$ is as $M$, that is, that the susceptibility of the instrument is directly as the bulk of the part of it immersed.

220. *Mr. Nicholson* has invented a form of the *Hydrometer* in which it may be applied to measure the specific gravities of *solid*, as well as *fluid* bodies.

Two metal cups $B$ and $C$, (Fig. 58.) are attached to a ball $A$, the lower $C$ being of sufficient weight to keep the instrument in a vertical position. When the Hydrometer floats in water let $\omega$ be the weight which must be placed in the upper cup $B$ to sink it to a given point $D$. This weight being replaced by the body whose specific gravity is required to be determined, let $\omega'$ be the weight which must be *added* in order again to sink the instrument to $D$. The body and weight $\omega'$ being now taken out of the upper cup, and the body placed in the lower, let $\omega''$ be the weight in $B$ requisite a third time to sink the instrument to $D$.

Call $W$ the weight of the instrument, $W'$ that of the fluid it displaces when immersed to $D$, $M$ the volume, $D$ the density of the body, and $D'$ the density of water.

Then since, in the two first cases the weights *added* to the instrument together with the weight of the instrument are respectively equal to the weight of the fluid displaced by the instrument, and in the third case, to the weight of the fluid displaced by the body *and* instrument, it follows that

$$W + \omega = W',$$

$$MDg + \omega' + W = W',$$

$$MDg + \omega'' + W = W' + MD'g.$$

Subtracting the first equation from the second, and the second from the third, we have

$$MDg = \omega - \omega',$$

$$\text{and } MD'g = \omega'' - \omega';$$

$$\therefore s = \frac{D}{D'} = \frac{\omega - \omega'}{\omega'' - \omega'}.$$

THE HYDROSTATIC BALANCE.

221. The Hydrostatic balance is a contrivance for determining the specific gravities of solid bodies by observing the weights lost by their immersion in fluids of known specific gravity. In its simplest form it may be described as a common balance in one of the scales of which the body is first weighed, and then, being suspended by a slender thread beneath it, again weighed when plunged in the given fluid. If $\omega$ and $\omega'$ be the weights necessary to produce equilibrium in the two cases, $\omega - \omega'$ is the weight lost in the immersion, and is therefore equal to that of the fluid displaced.

Calling therefore $M$ the mass of the body, $D$ its density, and $D'$ that of the fluid,

$$gMD = \omega,$$

$$gMD' = \omega - \omega';$$

$$\therefore \sigma = \frac{D}{D'} = \frac{\omega}{\omega - \omega'};$$

where $\sigma$ is the ratio of the specific gravities of the solid and fluid. If the fluid in which the immersion takes place be water,

$$s = \frac{\omega}{\omega - \omega'}.$$

222. The equation $gMD = \omega$, is only true on the hypothesis that the weight $\omega$ is determined *in vacuo*, or that, the body being of considerable density, its weight is exceedingly great as compared with that of an equal bulk of the air in which it is weighed.

To solve the problem with accuracy, allowance must be made for the buoyancy of the atmosphere. Let $D''$ be its density, $\omega$ the true weight of the body *in vacuo*, $\omega'$ in water, and $\omega''$ in air;

$$\therefore gDM = \omega,$$

$$gD'M = \omega - \omega',$$

$$gD''M = \omega - \omega'';$$

$$\therefore gM(D - D') = \omega'; \; gM(D'' - D') = \omega' - \omega'';$$

$$\therefore \frac{D - D'}{D'' - D'} = \frac{\dfrac{D}{D'} - 1}{\dfrac{D''}{D'} - 1} = \frac{\omega'}{\omega' - \omega''}.$$

Calling, therefore, $s$ the specific gravity of the body, and $s'$ that of air,

$$\frac{s - 1}{s' - 1} = \frac{\omega'}{\omega' - \omega''}; \quad \therefore s = \frac{\omega' s' - \omega''}{\omega' - \omega''}.$$

A correction, similar to the above, must be applied to all the cases in which the weight of the body is not determined *in vacuo*. Since the specific gravity of the atmosphere is variable, it is manifestly more accurate to weigh the body

in any other medium of known and permanent specific gravity. The same formula is applicable *mutatis mutandis*.

### THE HYDROSTATIC BELLOWS.

223. This instrument presents an illustration of what is termed the Hydrostatical paradox. It consists of two circular boards, $A$ and $B$, (Fig. 59.) firmly bound together by a cylindrical coating of leather or other pliable substance. $MN$ is a tube communicating with the lower portion of this cylinder.

Water being poured into the tube $MN$, the boards $A$ and $B$ will separate. $B$ will rise, and a weight $W$, which is exceedingly great as compared with that of the fluid in $MN$, may thus be supported at $B$. The fluid in $MN$, which is thus effective in supporting the weight $W$, is manifestly that $PN$, which is above the level of $B$. Calling $k$ the section of the tube $MN$, and $K$ the area of the board $B$, we have

$$\frac{\text{pressure on } P}{W} = \frac{k}{K}.$$

Let $M$ be the volume of the weight $W$, $D$ its density, and $D'$ that of water;

$$\therefore \frac{gD'.\overline{PN}.k}{gD.M} = \frac{k}{K};$$

$$\therefore PN = \frac{D}{D'} \frac{M}{K} = \frac{sM}{K},$$

where $s$ is the specific gravity of the weight $W$.

224. If $M'$ be the whole volume of fluid contained in the instrument,

$$AB(K+k) + \frac{Msk}{K} = M';$$

$$\therefore AB = \frac{M'K - Msk}{K(K+k)}.$$

If the quantity of fluid $M'$ contained in the vessel be increased by $m$, the corresponding ascent $B'$ of the weight $W$ will be represented by

$$\frac{K(M'+m)-Msk}{K(K+k)}-\frac{KM'-Msk}{K(K+k)}, \text{ or by } \frac{m}{K+k}.$$

### CECIL'S LAMP.

225. The oil is in this lamp kept continually at the *same height*, and thus supplied in the same quantity to the burner.

$ABD$ and $ACD$ (Fig. 61.) are two hemispheres, of which $ACD$ is solid and moveable in $ADB$, which is hollow, and has its interior diameter the same with the external diameter of the other. The oil is placed in the unoccupied portion $MNB$ of the hemisphere $ADB$, and as it consumes, its surface is kept continually at the same height by the revolution of $CAD$.

Let $P$ be the center of pressure and $Q$ the center of gravity of the plane $MND$, and $G$ the center of gravity of $CAD$. And let $OD=a$, $\angle BOD=\theta$; therefore (Art. 42. Ex. 3.)

$$OP=\frac{3}{16}\pi a, \quad OQ=\frac{4}{3}\frac{a}{\pi};$$

therefore pressure on $MND$

$$=\overline{QL}\,.\,\overline{MND}\,.\,Dg=\frac{4}{3}\frac{a}{\pi}\sin\theta\,.\,\frac{\pi a^2}{2}\,.\,Dg=\frac{2}{3}a^3\,.\,Dg\sin\theta;$$

therefore mom$^{m}$. of pressure upon $MND$

$$=\frac{3}{16}\pi a\frac{2}{3}a^3 Dg\sin\theta=\frac{1}{8}a^4 Dg\sin\theta.$$

Also $OG=\frac{3}{8}a$; therefore mom$^{m}$. of weight of $CAD$

$$=\overline{CAD}\,.\,\overline{OK}gD'=\frac{2}{3}\pi a^3\,.\,\frac{3}{8}a\sin\theta\, gD'=\frac{1}{4}a^4 D'g\sin\theta;$$

therefore there will be an equilibrium if

$$\frac{1}{8}a^4 Dg \sin\theta = \frac{1}{4}a^4 D'g \sin\theta;\ \text{or if } D' = \frac{1}{2}D.$$

And this expression being independent of $\theta$, it appears that, if the specific gravity of the solid hemisphere be half that of the fluid, the equilibrium will obtain in every position of the former.

### THE DIVING BELL.

226. *The Diving Bell* is commonly a hollow cylinder or parallelopiped, one end of which is closed. It is immersed with the open end downwards, weights, if necessary, being added to sink it and keep it steady in its descent. As the vessel descends, the fluid continually exercises a greater pressure on the contained air, condenses it, and occupies a greater portion of the vessel.* The bell being constructed of such dimensions in reference to the depth to which it is to be sunk, that a sufficient portion may remain unoccupied by the fluid; a platform is erected in this portion, on which those who descend take their station.

227. Let $x$ be the height of that portion of the bell which is free from water when the top is sunk to the depth $H$. Now, if $h$ be the height of an homogeneous atmosphere, $a$ the height of the bell, $D$ the density of the external air, $D_{,}$ that of the air in the bell, and $D'$ that of water,

$$g\{hD + (H+x)D'\}$$

will represent the pressure on an unit of the surface of the air contained in the bell. Also the unit of pressure arising from the elasticity of the internal air is $ghD_{,}$;

$$\therefore\ g\{hD + (H+x)D'\} = ghD_{,}$$

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* The method now commonly adopted is to form a communication between the bell and the surface, by means of a leathern pipe through which the air is continually forced by a condensing pump, and the surplus escaping under the edges of the bell ascends through the water. The air is thus continually changed and made fit for respiration, and the bell is kept entirely free from water.

But the *quantity* of air in the bell being the same as before immersion, and the section of it the same throughout,

$$D_{,}x = Da; \quad \therefore\ hDx + (Hx + x^2)D' = hDa.$$

Or calling $s$ the specific gravity of air,

$$x^2 + (H + hs)\,x = hsa;$$

$$\therefore\ x = -\tfrac{1}{2}(H + hs) \pm \sqrt{\tfrac{1}{4}(H + hs)^2 + hsa}.$$

### HERO'S FOUNTAIN.

228. $ABCD$ (Fig. 62.) is an air-tight vessel *connected* with another vessel $MKN$ and *sustained above it*, by tubes $DM$ and $NC$, of which, $DM$ reaches nearly to the bottom of $NK$ and passing through $AC$ communicates with an open reservoir above it, and $NC$ proceeding from the top of $KN$ reaches nearly to the top of $AC$. $GF$ is a tube proceeding from a point near the bottom of $AC$, and communicating with the external air by a small aperture at $F$.

Suppose the vessel $ADCB$ to be filled with water to the height of the extremity of the tube $NC$. And let water also be poured into the open reservoir which is above $ABCD$, and communicates with the tube $AK$. The fluid will descend through the tube $AK$ and occupy a portion of the vessel $MKN$, compressing the air in that vessel with a force equivalent to the weight of a column of water of the same base with the surface of the fluid, and of the altitude $AK$. The air thus compressed in the vessel $MKN$, in the tube $NC$, and above the surface of the fluid in the vessel $AC$, will exert on the latter surface a pressure, whose unit is greater than the unit of the pressure (of the external air) on the surface of the fluid within the tube $FG$, by the weight of a column equal to the height of the surface of the fluid in the reservoir above that in $NK$. Hence it appears, that the fluid will be projected through the orifice at $F$, and raised to the height $AV$ above the surface of that in $AC$.

229. Let $P$ and $Q$ be any positions of the surfaces of the fluid in the two vessels. Let $NQ = z$, $PC = z_{,}$, $AM = a$,

$K$ and $K_{,}$ sections of the vessels, and $k$ of the aperture $F$, $M$ the content of the vessel $NK$, $m$ that of the tube $CN$, and $t$ the time from the beginning of the motion. Now, the air at present occupying $AP$, $MQ$, and the tube $CN$, before the motion and under the pressure, occupied the space $M+m$;

$$\therefore \frac{h+a+z}{h} = \frac{M+m}{Kz+K_{,}z_{,}+m} \text{............(1)}.$$

Also the pressure at $F$ is equal to that of the column $AV$, less the column $FP'$;

$$\therefore \text{ velocity of projection } = \sqrt{2g(a+z-z_{,})};$$

$$\therefore t = \frac{K_{,}}{k\sqrt{2g}} \int \frac{dz_{,}}{(a+z-z_{,})^{\frac{1}{2}}} \text{........(2)}.$$

Eliminating $z$ between these equations, and integrating; $z_{,}$ and $z$, and therefore the velocity of projection and the height of the fountain may be found in terms of $t$.

## THE COMMON PUMP.

230. $AB$ and $BD$ (Fig. 63.) are cylinders connected together as in the figure. At $B$ is a valve $\mu'$ closing the lower cylinder and opening into the upper. $M$ is a piston accurately fitting the interior surface of the cylinder $AB$, and moveable along it by means of a rod $E\mu$ and a lever $EF$. In the center of the piston is an aperture $\mu$, closed by a valve which opens upwards. The cylinder $AB$ is terminated by a reservoir $AF$. The instrument is placed vertically, the suction pipe $BD$ being immersed in the fluid intended to be raised.

To explain the action of the pump, conceive the piston $M$ to be depressed to $B$, the air in $MB$ forcing up the valve $\mu$ and escaping through the aperture. The piston $M$ being then at $B$ and both valves closed, suppose it to be drawn up again to $A$. By the ascent of the piston $M$, the

valve $\mu$ remaining closed, and no air being suffered to enter between the surface of the cylinder and the edges of the piston, the pressure of the air in $AB$ on the *upper* surface of the valve $\mu'$ will be removed. The pressure, therefore, upon its *inferior* surface, arising from the elasticity of the air in $BC$, being no longer counteracted by an equal and opposite pressure upon its *superior* surface, the valve will open, and the air in $BC$ expand itself over the whole space $AC$. Its density and therefore its elasticity will thus be diminished, and the pressure on the surface of the fluid *within* the tube $BC$ will become *less* than that of the surrounding atmosphere on an equal portion of the surface $C'C''$ of the fluid *without* it. The equilibrium of the surface $C'C''$ therefore will be destroyed, and the fluid will ascend in the tube $BC$, until the weight of the column $CP$ above $C$ is such, that the pressure on the section within the tube at $C$ is the same with that on an equal portion of the surface without it.

Let the piston be now again made to descend from $A$ to $B$*. The air in $AB$, by the continual contraction of the space in which it is contained, will again be brought to a density greater than that of the external air, the valve $\mu$ will therefore again be forced open, the air beneath the piston will escape, and by its re-ascent the air in $BP$ will be still farther rarified by expansion over the space $AP$.

The equilibrium will thus again be destroyed, and restored by a farther ascent of the surface $P$. And thus by repeated strokes of the piston the fluid may be raised to the level of $B$, and made to pass through the valve $\mu'$ into the barrel of the pump $BA$. This being once effected, at each descent of the piston a portion of the fluid will force itself through the valve $\mu$ into the space above it, and at each ascent will be raised into the reservoir $AF$, and discharged through the spout.

231. Let $P$ be the position of the surface of the fluid after the $n^{\text{th}}$ stroke of the piston, $h_n$ the height of a column

* The air in $ABP$ being of the same uniform density, the valve $\mu'$ will close by its own weight.

of water which would at this time be supported by the elasticity of the air in $BP$, and let $BP = x_n$; also let $h_{n+1}$, $x_{n+1}$ be quantities similarly taken after the $(n+1)^{\text{th}}$ stroke. Let $h$ be the height of a column of water whose pressure is equal to that of the external atmosphere, $AB = a$, $BC = b$, $K =$ section of $AB$, $k =$ section of $BC$. Then since the elasticity of the air in $BP$ together with the weight of the column of water $PC$ is equal to the pressure of the atmosphere on an equal portion of the external surface $CC'$; we have

$$h_n + (b - x_n) = h;$$

$$\therefore\ h_n - x_n = h - b,$$

and this is true for all values of $n$;

$$\therefore\ h_{n+1} - x_{n+1} = h - b.$$

Again, since $h_n$ and $h_{n+1}$ are as the densities of the air in the pump after the $n^{\text{th}}$ and $(n+1)^{\text{th}}$ strokes, and that by the $(n+1)^{\text{th}}$, the air in $BP$ is expanded over the space $AQ$, ($Q$ being the position of the surface after that stroke); we have

$$h_{n+1}\left\{Ka + kx_{n+1}\right\} = kh_n x_n.$$

Therefore eliminating $x_{n+1}$,

$$h_{n+1}\left\{Ka + k(h_{n+1} - h + b)\right\} = kh_n x_n;$$

$$\therefore\ h_{n+1} = -\tfrac{1}{2}\left\{\frac{Ka}{k} + b - h\right\} \pm \tfrac{1}{2}\sqrt{\left\{\frac{Ka}{k} + b - h\right\}^2 + 4h_n x_n};$$

$$\therefore\ x_{n+1} = -\tfrac{1}{2}\left\{\frac{Ka}{k} + h - b\right\} \pm \tfrac{1}{2}\sqrt{\left\{\frac{Ka}{k} + b - h\right\}^2 + 4h_n x_n}.$$

232. The length of the pipe $BC$ is necessarily less than $h$, since, otherwise, before the fluid has risen to $B$, the weight of the column $PC$ will be equal to the atmospheric pressure on an equal portion of the surface $C'C''$, and could even a vacuum be produced in $MP$, no further ascent of the fluid would follow.

233. In practice, the valves cannot be made accurately air-tight, nor can the piston be made accurately to fit the interior of the barrel, or to descend entirely to its lower extremity $B$. This last cause may entirely arrest the action of the pump. Suppose the play of the piston to be through the distance $a'$ instead of $a$, or from $A$ to $B'$.

At every *descent* of the piston, the air in $AB$ will be compressed into the space $BB'$. Now it is necessary that its elasticity when thus condensed should be greater than that of the external atmosphere, since otherwise the valve $\mu$ will not be raised and no further rarefaction can take place beneath it. Now the elasticity of the air in $AB$ after the $n^{\text{th}}$ *ascent* is sufficient to support a column of water $h_n$. If therefore $h_n'$ represent similarly the elasticity of that in $BB'$, after the $n^{\text{th}}$ *descent* of the piston, since the elasticity of air is as its density,

$$K(a - a')\,h_n' = Kah_n;$$

$$\therefore\ h_n' = \frac{ah_n}{a - a'}.$$

Now the column of water supported by the external air is $h$. Therefore in order that the action of the pump may continue after the $n^{\text{th}}$ stroke, $h_n'$ must exceed $h$,

$$\text{or } \frac{ah_n}{a-a'} > h \text{ or } h_n > \left(1 - \frac{a'}{a}\right)h.$$

To determine the height at which the ascent will cease, we have, since $h_n - x_n = h - b$,

$$x_n + h - b > \left(1 - \frac{a'}{a}\right)h; \quad \therefore\ b - x_n < \frac{a'h}{a}.$$

If therefore $b$ be not less than the quantity $\frac{a'}{a}h$ water cannot be raised into the barrel of the pump.

234. If $CN = h$, it is clear that the water entering the barrel at each ascent of the piston cannot rise *above* $N$. In order therefore that the pump may discharge at every stroke

of the piston its *maximum quantity* of fluid the distance of the extreme ascent of the piston above $C$ must *exceed* the quantity $h$. If the surface $C'C''$ sink continually with the discharge, the quantity of water raised at each stroke will begin to diminish when $AC$ becomes less than $h$, and the discharge will wholly cease when $BC$ is less than $h$.

235. To determine the height $H$ of a column of water whose weight is equivalent to the pressure on the piston $M$, when the surface of the fluid is at any point $N$ above it, we have: pressure of fluid in $CM$ *upwards* on $M$ = the weight of the column $h - CM$: pressure *downwards* on $M$ = weight of column $H + NM + h$;

$$\therefore NM + h - H = h - CM;$$

$$\therefore H = CM + NM = CN.$$

## ARCHIMEDES' SCREW.

$ABC$ (Fig. 66.) is a tube wound in a spiral direction round a cylinder, which is moveable about its axis, and inclined at a given angle to the horizon. The extremity $A$ is immersed in a fluid which fills the portion of the tube $AA'$, beneath its surface.

The cylinder is put in motion about its axis, so that the extremity $A$ and the portions $B$, $C$, &c. represented in the figure as nearest to the eye, may ascend whilst $A'$, $B'$, $C'$, &c. on the opposite side of it, descend.

Now by the nature of fluid equilibrium, the surfaces $A$ and $A'$ tend continually to establish themselves in the same horizontal plane. The surface $A$ being therefore raised, and $A'$ depressed, by the motion of the cylinder, it is clear that the former will descend *along the tube* towards $A'$ and the latter ascend towards $B$, and thus a continual motion will be produced along the tube towards $D$. After the first revolution of the cylinder; the fluid in $AA'$ will be made to occupy the position $BB'$, and the extremity $A$ being brought again to the surface of the fluid

to be raised, the portion $AA'$ of the tube will be a second time filled. By the second revolution this portion of fluid will be transferred to $BB'$, and that in $BB'$ to $CC'$. And thus eventually all the different portions of the screw similarly taken will be filled with columns of fluid, which will continually be made to move along it, and successively discharged at its extremity. $AA'$, $BB'$, &c. are called the hydrophorous arcs of the screw.

To find the equations to the screw of Archimedes.

Let the axis of the cylinder lie in the plane of $xz$, (Fig. 67.) then the base $ACO$ of the cylinder is perpendicular to that plane. From $P$, a point in the curve, draw $PC$ parallel to the axis of the cylinder, and $CD$, $Dd$, respectively parallel to the axes $Oy$, $Ox$. Let $AC=\theta$, therefore $CP=\theta \tan a$; if $a=$ the constant angle at which every element of $AP$ is inclined to $ACO$. Let $\angle AOx'=e$;

$$\text{then dist. of } P \text{ from the plane of } xz = CD = \sin\theta;$$

$$\text{dist. of } C \text{ from the plane of } zy = Dd = (1+\cos\theta)\cos e,$$

and therefore, dist. of $P$ from that plane

$$= CP \sin e - Dd = \theta \tan a \sin e - (1+\cos\theta)\cos e;$$

$$\text{dist. of } D \text{ from the plane of } xy = Od = (1+\cos\theta)\sin e;$$

and therefore, dist. of $P$ from that plane

$$= CP \cos e + Od = \theta \tan a \cos e + (1+\cos\theta)\sin e.$$

If therefore, $x$, $y$, $z$, be co-ordinates of $P$ measured from $O$, parallel to the axes $Ox$, $Oy$, $Oz$,

$$\text{then } x = \theta \tan a \sin e - (1+\cos\theta)\cos e,\ \ y=\sin\theta,$$

$$z = \theta \tan a \cos e + (1+\cos\theta)\sin e \ldots\ldots\ldots (1);$$

and the equations are

$$x = \sin^{-1}y \tan a \sin e - (1+\sqrt{1-y^2})\cos e,$$

$$z = \sin^{-1}y \tan a \cos e + (1+\sqrt{1-y^2})\sin e.$$

Cor. The value of $\theta$, corresponding to that point in the screw whose height above the plane of $xy$ is a maximum, is given by the equation $\sin\theta = \frac{\tan a}{\tan e}$; as appears by making $\frac{dz}{d\theta} = 0$, in equation (1).

### THE AIR PUMP.

236. $AB$ and $A'B'$ (Fig. 48.) are cylinders commonly of the same size, in which pistons $P$ and $Q$ are moveable alternately by means of rack work, the one ascending whilst the other descends. At the inferior extremities $A$ and $A'$ of the cylinders, are valves opening downwards. And at $B$ and $B'$ are apertures communicating with a vessel from which the air is to be exhausted, and which is called the receiver.

By the ascent of the piston $P$ a vacuum is produced in the space $AP$ beneath it, the valve $A$ closing by the pressure of the external air. When the piston has ascended above the aperture $B$, a communication being opened between the receiver and the vacuum $AP$, the air from the former rushes into the latter space, and the whole acquires the same uniform density. On the return of the piston $P$, the air thus occupying the space $AB$ is condensed until it acquires sufficient elasticity to force open the valve $A$, when it is expelled, and a second exhaustion takes place by the re-ascent of the piston. The action of both pistons is manifestly the same, and thus for each descent of either piston a volume of air is expelled from the machine equal to the content of either cylinder.

237. Let $M$ be the content of the whole machine, receiver, tubes, and cylinders. Let $N$ be the content of either cylinder or barrel, and let $D_n$ be the density of the air contained in the machine after the $n^{\text{th}}$ stroke. Now $D_nM$ is the quantity of air contained in the machine after the $n^{\text{th}}$ stroke, and the volume $N$ of it, or the quantity $D_nN$, is expelled by the $(n+1)^{\text{th}}$ stroke; there remains therefore in the machine the quantity of air $D_nM - D_nN$ after the $(n+1)^{\text{th}}$ stroke. And this is expanded over the space $M$;

$$\therefore\ D_{n+1} M = D_n M - D_n N;$$

$$\therefore\ D_{n+1} = D_n \left\{1 - \frac{N}{M}\right\};$$

$$\therefore\ D_n = D \left\{1 - \frac{N}{M}\right\}^n;$$

if $D$ represent the initial density of the air.

## THE CONDENSER.

238. $AB$ (Fig. 49.) is a cylinder, communicating with a receiver $M$, and $P$ a piston moveable in it, in which and at $B$, are valves opening downwards.

The piston $P$ being forced down the barrel, the valve in it is closed by the elasticity of the air condensed beneath it, whilst the valve at $B$ is forced open. And when the piston has arrived at $B$, the whole of the air in $AB$ has been forced into $M$. On the return of the piston the valve $B$ closes by reason of the excess of the elastic force of the air in $M$ over that of the external air, admitted, through the valve in $P$, into the space $BP$. No air therefore escapes from $M$, and at every descent of the piston a quantity of air equal in volume to the content of the barrel, and of the same density with the external air, is forced into it.

239. If therefore $D_n$ and $D_{n+1}$ be the densities of the air in $M$ after the $n^{\text{th}}$ and $(n+1)^{\text{th}}$ descents, $M$ its content and $N$ that of the barrel, also $D$ the density of the external air, then

$$D_{n+1} . M = D_n . M + D . N;$$

$$\therefore\ D_{n+1} - D_n = \frac{N}{M} . D;$$

$$\therefore\ D_n = D \left\{1 + \frac{Nn}{M}\right\}.$$

240. If the piston do not descend immediately to the bottom of the barrel, and the space intervening between its extreme descent and the receiver, be $m$,

$$D_{n+1}(M+m) = D_n . M + D . N;$$

$$\therefore\ D_{n+1} - \left(\frac{M}{M+m}\right) D_n = \left(\frac{DN}{M+m}\right),$$

by the solution of which equation of finite differences we obtain,

$$D_n = D \cdot \left(\frac{M}{M+m}\right)^n \cdot \left\{ \Sigma \frac{N}{M} \left(1 + \frac{m}{M}\right)^n + 1 \right\} - \frac{ND}{M}.$$

The above solution applies to the case in which a given *volume* of air, $m$, is supposed at each ascent to escape from the condenser whilst the valve $B$ is closing.

### THE BAROMETER.

241. A glass tube $AB$ (Fig. 47.) having its extremity $A$ hermetically sealed, is *filled* with mercury and inverted in a vessel of the same fluid. Now it is necessary to the equilibrium of a heavy fluid that the unit of pressure on *every portion* of a horizontal section any where made in it should be the same. The surface $P$ of the fluid in the tube will therefore descend* until the unit of pressure on the horizontal plane $CC'$ is the same within and without the tube. Now within the tube, it is the weight of the superincumbent column of mercury $CP$, and without, it is the weight of the superincumbent column of air. The weight of the mercury in the tube of the barometer, is therefore a measure of the weight of a column of the atmosphere extending from the place of observation to its surface. Also this last weight is proportional to the elasticity and density of the air at the place of observation.

* The length of the tube is here supposed to *exceed* the height of a column of mercury equivalent in weight to the atmospheric pressure.

Now if the *density* of the mercury in the barometer were always the same, its *weight* would vary as its *height* in the tube, and this *height* would be an accurate measure of the weight, elasticity, and density of the atmosphere. This is however by no means the case. The *density* of mercury in common with that of all other bodies varies with any variation in the temperature.

242. It is found by experiment that mercury expands or contracts by equal fractions of its bulk for all equal increments or decrements in its temperature. And that for the variation of one degree as measured by the centigrade thermometer, the corresponding fraction is $\frac{1}{5412}$. Hence if $V$ be the volume of any given quantity of mercury at the temperature $T^0$; then at the less temperature $T'^0$, this volume will have been diminished by the quantity $\dfrac{T^0 - T'^0}{5412} \cdot V$. And if $D$ and $D'$ be the corresponding densities, since the quantity of the fluid is the same,

$$DV = D'\left\{V - \frac{T^0 - T'^0}{5412} V\right\}; \quad \therefore\ D' = \frac{D}{1 - \dfrac{T^0 - T'^0}{5412}}.$$

Hence if $H'$ represent the height of the mercury at the temperature $T'^0$, the weight of a column whose base is unity is represented by

$$D'H'g = \frac{DH'g}{1 - \dfrac{T^0 - T'^0}{5412}}.$$

Also, if *density* $(T'^0, H')$, represent the density of the air at the temperature $T'$ of the mercury and height of barometer $H'$, we have

$$\textit{density}\,(T'^0, H') = \frac{CDH'g}{1 - \dfrac{T^0 - T'^0}{5412}},$$

similarly *density* $(T^0, H) = CDHg$;

$$\therefore \frac{density\ (T'^0, H')}{density\ (T^0, H)} = \frac{pressure\ (T'^0, H')}{pressure\ (T^0, H)} = \frac{\left(\frac{H'}{H}\right)}{1 - \frac{T^0 - T'^0}{5412}}.$$

243. We have here supposed the force of gravity to be the *same* at the places of observation. If, however, the observations be made at different distances from the Earth's center, this will not be the case. Suppose one made at the Earth's surface, and the other at the altitude $z$ above it;

$$\therefore density\ (T'^0, H', z) = \frac{CH'Da^2g}{\left(1 - \frac{T^0 - T'^0}{5412}\right)(a+z)^2},$$

$$density\ (T, H,)\ 0 = C\,.\,HDg;$$

$$\therefore \frac{density\ (T'^0, H', z)}{density\ (T^0, H, 0)} = \frac{pressure\ (T'^0, H', z)}{pressure\ (T^0, H, 0)} = \frac{\left(\frac{H'}{H}\right)}{\left(1 - \frac{T^0 - T'^0}{5412}\right)} \cdot \left(\frac{a}{a+z}\right)^2$$

244. *To determine the heights of mountains by means of barometrical observations.*

By Article 125, we have, if $p$ and $p'$ represent the units of atmospheric pressure at the Earth's surface, and at the altitude $z$ above it,

$$\text{h. l.}\left(\frac{p}{p'}\right) = \frac{cagz}{\left\{1 + \frac{t+t'}{500}\right\}(a+z)};$$

therefore by the last article,

$$\text{h. l.}\left\{\frac{\left(1 - \frac{T^0 - T'^0}{5412}\right)}{\left(\frac{H'}{H}\right)}\left(\frac{a+z}{a}\right)^2\right\} = \frac{cagz}{\left\{1 + \frac{t+t'}{500}\right\}(a+z)}.$$

Whence we obtain

$$z = \left\{\frac{1+\frac{t+t'}{500}}{cg}\right\} \left\{\text{h. l.} \frac{\left(1-\frac{T-T'}{5412}\right)}{\left(\frac{H}{H'}\right)} + 2\,\text{h. l.} \left(1+\frac{z}{a}\right)\right\} \left(1+\frac{z}{a}\right).$$

245. The constant $\frac{1}{cg}$ may be determined by observing the values $H$, $H'$, $T$, &c. for some known value of $z$. It has been ascertained by numerous observations of this kind to be 10050 fathoms in latitude $50^0$. The quantity $g$ is however *variable* with the latitude. Generally for the latitude $\lambda$,

$$\frac{1}{cg} = (10050) \left\{1 + \cdot 002837 \cos 2\lambda\right\};$$

$$\therefore z = (10050) \left\{1 + \cdot 002837 \cos 2\lambda\right\} \left\{1 + \frac{t+t'}{500}\right\}$$

$$\left\{\text{h. l.} \left(\frac{1-\frac{T-T'}{5412}}{\frac{H}{H'}}\right) + 2\,\text{h. l.} \left(1+\frac{z}{a}\right)\right\} \left(1+\frac{z}{a}\right).$$

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246. *To find the altitude of the mercury in the barometer when a portion of air has been allowed to remain in the upper part of the tube.*

Let the air in the tube, when of the same density with the external air, occupy the space $AQ$, (Fig. 47.) and let the mercury stand at $P$, $AB=a$, $AQ=b$, $AP=x$, $h=$ the height at which the mercury would stand if a vacuum were produced above it, $h'=$ the height of the column of mercury which would be sustained by the elasticity of the air in $AP$. Now the densities in $AQ$ and $AP$ are as $h$ and $h'$. Also the *quantity* of air in each is the same;

$$\therefore hb = h'(a-x)$$

Now the weight of the column $BP$, and the elastic pressure of the air in $AP$ on its surface, are sustained by the pressure of the external air;

$$\therefore\ h = h' + x;$$

$$\therefore\ \frac{hb}{h-x} = a - x.$$

Whence $x$ may be found, or if $x$ be observed, the true height $h$ of the barometer may be found.

THE SEA GAGE.

247. $AB$ (Fig. 46.) is a vessel perforated with holes, within it is firmly fixed, in a vertical position, a glass tube, having one end hermetically closed, and the other immersed in a cup of quicksilver. $A$ is a hollow sphere, whose buoyancy is sufficient to raise the instrument, when a weight $W$ hung at the bottom of it is detached. The instrument is allowed to sink in the water whose depth is to be determined, and there is a mechanical contrivance by means of which, when it strikes the bottom, the weight $W$ is detached, and the gage made to re-ascend by the buoyancy of the ball. The height to which the mercury has been made to ascend in the tube, is marked by the adhesion of oil or any other viscid substance, placed on its surface, to the interior of the tube.

248. Let $h'$ be this height ($MP$), and $h$ the height of the barometer at the surface, and $x$ the depth of the fluid, $l$ the length of the tube above the surface of the mercury in the cup. Now the column of mercury which will be sustained by the elasticity of the air is as its density. Let $h''$ be the height of the column which would be sustained by the elasticity of the air in $NP$;

$$\therefore\ h''(l-h') = hl.$$

*Also the elasticity of the air in* NP + *the weight of the column* MP = *pressure of the atmosphere on the surface*

*of fluid + the weight of a column of water extending to the bottom.*

$$\therefore\ Dh'' + Dh = Dh + D'x,$$

where $D$ = the density of mercury and $D'$ of water;

$$\therefore\ x = \sigma h'' = \frac{\sigma h l}{l - h'} = \frac{\sigma h}{1 - \left(\frac{h'}{l}\right)};$$

whence it appears that if the tube be divided into $m$ equal parts, and the mercury ascend to the $n^{\text{th}}$ division of the scale

$$x = \frac{\sigma h}{1 - \frac{n}{m}}.$$

## CLEPSYDRA.

249. The clepsydra or water clock is a contrivance for marking the time by the descent of the surface of a fluid which flows through a small aperture in the base of the vessel which contains it.

Suppose the vessel a prism. It is required to determine what scale must be marked on its side, that the coincidence of the descending surface with the successive lines of the division may mark equal successive intervals of time.

Let $x$ be the distance of the surface from the base of the vessel at the end of any time $t$, from the beginning of the motion; when let the value of $x$ have been $a$.

Let $K$ be a horizontal section of the prism, and $k$ of the aperture;

$$\therefore \text{ by Art. 143. } x = a - \frac{k\sqrt{2ga}}{K}t + \frac{k^2 g}{2K^2}t^2.$$

Let $\Delta x$ and $\Delta t$ be corresponding increments of $x$ and $t$;

$$\therefore\ x + \Delta x = a - \frac{k\sqrt{2ga}}{K}(t + \Delta t) + \frac{k^2 g}{2K^2}(t + \Delta t)^2;$$

whence we obtain

$$\Delta x = -\frac{k^2 g}{K^2}\left\{\frac{K}{k}\sqrt{\frac{2a}{g}} - t - \Delta t\right\}\Delta t.$$

The time $t$ is of course in seconds. To determine the divisions corresponding to successive *minutes* of time, write for $\Delta t$, $60''$ and give to $t$ the successive values 0, $60''$, $120''$, &c. If $\frac{K}{k}\sqrt{\frac{2a}{g}} - \Delta t = 0$, the distance of the divisions will vary as the time.

250. To find the form of a clepsydra such that the whole descent may vary as the time from rest, or the surface descend through equal distances in equal successive intervals of time.

Let $AM$ (Fig. 60.) be the axis of the clepsydra, and $PQ$ any position of the surface. Then since the vessel is regular and symmetrical about $AM$, each section $PQ$ varies as $PM^2$.

Let $AM = x$, $PM = y$, and let the section $PQ = cy^2$;

$$\therefore\ kvdt = -cy^2 dx.$$

Now if $a$ be the height from which the surface has descended,

$$\text{by hypothesis } t \propto a - x = b(a - x);$$

$$\therefore\ dt = -bdx \text{ and } \therefore\ kvb = cy^2;$$

$$\therefore\ kb\sqrt{2gx} = cy^2 \text{ and } y = \left(\frac{\sqrt{2g}.kb}{c}\right)^{\frac{1}{2}} x^{\frac{1}{4}}.$$

## THE COMPOUND FLOAT.

251. The compound float is used for comparing the velocities at different depths in a stream of fluid. $A$ and $B$ (Fig. 44.) are two spheres connected by a thread, of which $A$ is the lighter.

Being thrown into the stream, the balls will, after a certain time, have acquired a common velocity, and the float a permanent position. Let the common velocity $V$ be observed, call $v$

the velocity of the stream at $A$, and $v'$ that at $B$. Let $a$ and $\beta$ be the radii of the balls. Conceive the velocity $V$ to be communicated to the whole system (the float and fluid) in a direction opposite to that of the motion of the float. The action of the fluid upon the balls, and the relative position of the float and stream, will then remain precisely as before, and the float will be at rest. Now, resolving the forces which hold the float in equilibrium, in a horizontal and vertical direction, it is manifest that the latter destroy (since the system has no vertical motion) and that the former are the pressures of the *stream* on the balls $A$ and $B$. But at $A$ the stream is moving in the direction $A'A$ with the velocity $V-v$, and at $B$ in the direction $B'B$ with the velocity $v'-V$. The resistances of the balls to the stream, or the pressures of the stream on the balls, are therefore represented by $ca^2(V-v)^2$ and $c\beta^2(v'-V)^2$, (Art. 175.) And since these constitute the only horizontal forces impressed on the float, and that they are in opposite directions, we have

$$ca^2(V-v)^2 - c\beta^2(v'-V)^2 = 0;$$

$$\therefore \frac{v'-V}{V-v} = \frac{a}{\beta}; \quad \therefore v' = V\left(1+\frac{a}{\beta}\right) - \frac{av}{\beta}.$$

If the weight of the float be adjusted so that the ball $A$ may move in contact with the surface of the stream; the velocity $v'$ at any depth below it, may be determined in terms of that at the surface; this last being ascertained by the motion of a light body upon the current. Now in order that the ball $A$ may *just* float in contact with the surface of the fluid, we must have

$$\sigma a^3 + \sigma'\beta^3 = a^3 + \beta^3;$$

where $\sigma$ and $\sigma'$ are the specific gravities of the balls.

252. To find the *position* the float will assume when in equilibrium; taking $A$ for the origin, $\lambda$ for the length of the string, and $\theta$ for its inclination to the vertical; since the vertical force impressed on $B = \frac{4}{3}\pi(\sigma'-1)\beta^3 g$, and the horizontal force $= \frac{1}{4}\pi(v'-V)^2\beta^2$, we have, by the general equation of equilibrium, $\Sigma(Xy - Yx) = 0$,

$$\tfrac{1}{4}\pi (v' - V)^2\beta^2.\lambda.\cos\theta - \tfrac{4}{3}(\sigma' - 1)\beta^3 g\lambda\sin\theta = 0,$$

$$\therefore\ \tan\theta = \tfrac{3}{16}.\frac{(v' - V)^2}{\beta(\sigma' - 1)g}.$$

### PITOT'S TUBE.

253. The bent tube $ABP$ (Fig. 39.) open at both extremities, is plunged in the fluid to the depth at which the velocity is required to be determined. The arm $PB$ is kept in a vertical position, and $AB$ turned first in the direction of the motion of the fluid and then in a direction opposite to it.

In the former case, the surface $P$ of the fluid in the vertical tube $AP$ will be below, and in the other, above that of the stream. The weight of the fluid in $AP$ being in both cases equivalent to the pressure at $A$.

Calling $v$ the velocity and $z$ the depth; since in the one case the motion of the fluid contiguous to $A$ tends to increase the co-ordinates, and in the other to diminish them, (see Art. 171. Note);

$$g.\overline{BP} = gz - \tfrac{1}{2}v^2 \text{ and } g\overline{BQ} = gz + \tfrac{1}{2}v^2;$$

$$\therefore\ g\overline{PQ} = v^2 \text{ and } v = \sqrt{g\overline{PQ}}.$$

The distance $PQ$ being observed, the velocity is therefore known.

### HYDRAULIC QUADRANT.

254. $B$ (Fig. 45.) is a sphere suspended in a running stream by means of a string attached to a fixed point $A$. $AB$ is made to deviate from the vertical by the impulse of the stream, and the angle of deviation being observed, the velocity of the stream is thence deduced. Call the velocity $v$, the radius of the sphere $a$, and $AB$, $b$; $\therefore$ resistance on $B = \frac{1}{4}\pi a^2 v^2$. And its buoyancy or the vertical pressure of the fluid upon it is represented by $\frac{4}{3}\pi a^3(\sigma - 1)g$.

Therefore when the system is in equilibrium, we have by the general equation $\Sigma(Xy - Yx) = 0$, taking $A$ for the origin,

$$\tfrac{1}{4}\pi a^2 b v^2 \cos\theta - \tfrac{4}{3}\pi b a^3 (\sigma - 1) g \sin\theta = 0;$$

$$\therefore\ v = 4\sqrt{\tfrac{1}{3} a g (\sigma - 1)\tan\theta}.$$

255. If the string $BA$ pass over a *pulley* at $A$, and a weight $B$ be attached to its extremity; resolving the tension of $P$ on $B$ in a horizontal and vertical direction, we have to take, in addition to the former forces, the two $-P\sin\theta . g$ and $-P\cos\theta g$;

$$\therefore\ \tfrac{4}{3}\pi a^3 g(\sigma - 1) - P\cos\theta\, g = 0,$$

$$\text{and } \tfrac{1}{4}\pi a^2 v^2 - P\sin\theta g = 0.$$

These involve the preceding equation. Eliminating $\theta$,

$$v = 2\sqrt{g}\sqrt[4]{\frac{P^2}{\pi^2 a^2} - \frac{16}{9} a^2 (\sigma - 1)^2}.$$

From this last equation the velocity is known by simply observing the weight necessary to produce equilibrium.

## THE COMMON TUBE OR CONDUIT PIPE.

256. A fluid descends freely in the tube $AP$ (Fig. 38.) and escapes through its extremity $A$. $P$ is any position of its surface, $AP = s$, $AM = z$.

$$\therefore\ \text{(Art. 167.)}\ v^2 = -2g\int\frac{z\,ds}{s}.$$

Suppose the tube to be composed of a vertical and horizontal arm as in Fig. 39.

Let $AB = a$, $BP = z$;

$$\therefore\ v^2 = -2g\int\frac{z\,dz}{a+z} = -2g\int\left\{dz - \frac{a\,dz}{a+z}\right\}$$

$$= 2g\left\{z_1 - z - a \text{ h. l. } \frac{a+z_1}{a+z}\right\},$$

taking the integral from $z_1$ to $z$. If there be any number of elbows in the tube, similar to that at $B$; the same result will be obtained, $a$ representing the *sum* of the horizontal portions of the tube.

257. If the section ($K$) of the vertical portion $BP$ of the pipe be greater than that ($k$) of the other; adopting the hypothesis of parallel sections, and calling $z$ the height $BP$ of the fluid in the vertical tube, we shall have (Art. 159.)

$$v^2 = \frac{K^2}{k^2}\,\epsilon^{\left(\frac{\frac{K^2}{k^2}-1}{K}\right)\int\frac{dz}{N}} \int\left\{-2g\frac{z}{KN}\,\epsilon^{\left(\frac{\frac{K^2}{k^2}-1}{K}\right)\int\frac{-dz}{N}}dz + C\right\}.$$

Now, if $s$ be the length of the column of fluid, at any time contained in the tube $BA$,

$$N = \frac{z}{K} + \frac{s}{k}.$$

Also, if $m^2 =$ the whole quantity of fluid contained in the tubes $Kz + ks = m^2$;

$$\therefore \text{ eliminating } s, \quad N = \frac{m^2}{k^2} - \frac{K^2 - k^2}{Kk^2}z;$$

$$\therefore \frac{dN}{N} = \frac{K^2-k^2}{Kk^2}\,\frac{-dz}{N};$$

$$\therefore \text{h.l. } N = \left\{\frac{\frac{K^2}{k^2}-1}{K}\right\}\int\frac{-dz}{N};$$

$$\therefore v^2 = \frac{K^2}{Nk^4}\left\{\int\frac{-2gzdz}{K} + C\right\};$$

$$\therefore v^2 = \frac{K^2}{Nk^2}\left\{\frac{g(a^2-z^2)}{K}\right\} = \frac{gK^2(a^2-z^2)}{Km^2-(K^2-k^2)z},$$

taking the integral from $a$ to $z$.

If the fluid, instead of moving continually along the tube $BA$, escape from its extremity $A$; and the section $k$ be exceedingly small when compared with $K$, so that the surface $P$ may be considered stationary; we shall obtain from equation $B$, (Art. 158.)

$$0 = ga - k\frac{dv}{dt}N - \tfrac{1}{2}v^2,$$

where $N$ is constant and $= \frac{a}{K} + \frac{s}{k}$;

$$\therefore\ t = Nk\int\frac{dv}{ga - \frac{1}{2}v^2} = \frac{Nk}{\sqrt{2ga}}\ \text{h.l.}\left\{\frac{\sqrt{2ga}+v}{\sqrt{2ga}-v}\right\};$$

$$\therefore\ v = \sqrt{2ga}\left\{\frac{\epsilon^{\frac{\sqrt{2ga}}{Nk}t} - 1}{\epsilon^{\frac{\sqrt{2ga}}{Nk}t} + 1}\right\}.$$

The integral is taken, above, on the supposition that when $t = 0$, $v = 0$.

If the initial velocity be $v_1$

$$t = \frac{Nk}{\sqrt{2ga}}\ \text{h.l.}\left\{\frac{\sqrt{2ga}+v}{\sqrt{2ga}-v}\right\}\cdot\left\{\frac{\sqrt{2ga}-v_1}{\sqrt{2ga}+v_1}\right\}.$$

The *quantity* of fluid discharged

$$= \int kv\,dt = Nk^2\int\frac{v\,dv}{ga - \frac{1}{2}v^2} = Nk^2\ \text{h.l.}\left\{\frac{2ga - v_1^2}{2ga - v^2}\right\}.$$

If the pressure of the fluid be in a direction opposite to the motion, $a$ will become negative, and the quantity of fluid discharged will be represented by

$$Nk^2\ \text{h.l.}\left\{\frac{2ga + v_1^2}{2ga + v^2}\right\}.$$

## THE SYPHON.

258. If a bent tube $PQ$ (Fig. 40.) be filled with a fluid, and one extremity $P$ immersed in a vessel of the same fluid; then if the *other* extremity $Q$ be in the same horizontal plane with $A$, the fluid contained in the tube will remain at rest; if it be raised above this plane the fluid will flow back into the vessel; and if depressed below, it will escape through $Q$ and empty the vessel to the level of $P$. For the pressure *within* the tube at $P$ is equal to the weight of a superincumbent column of the fluid of the height $RM$; and that at $Q$ to the weight of a column $RN$. Also the *external* pressure on $P$ is the weight of the column of water $PA$, together with that of the superincumbent column of air, whilst that at $Q$ is only the weight of the superincumbent column of air. On the whole therefore, the column of fluid in the tube is thrust upwards at its extremity $P$ by a force represented by the weight of a column of atmosphere diminished by the weight of the column of fluid $RA'$; whilst at $Q$ it is similarly thrust upwards by the weight of the same column of incumbent air diminished by the weight of the column of water $RN$. If therefore (as in the figure) $RN$ be *greater* than $RA'$, the column will give way to the former pressure, and flow in a continual stream through $Q$. If $RN$ be *equal* to $RA'$, or $Q$ on the same level with $A$, the pressures at $P$ and $Q$ will be equal and the fluid will remain at rest. And if $RN$ be *less* than $RA'$, the pressure at the extremity $Q$ will be the greater, and the fluid will flow back into the vessel. The moving force in all cases is the weight of the column $A'N$.

The water in the syphon and vessel forms one continued mass, held together by the atmospheric pressure on the surfaces $A$ and $Q$, so long as the weight of the column $RN$ does not exceed that pressure. When this is the case, there is, in fact, no effective pressure on the column $QR$ upwards, and it *may* separate itself from $RP$. If the weight of the column $RA'$ *also* exceed the atmospheric pressure, no force will tend to press the column $PR$ upwards; and the two $RP$ and $RQ$ *must* separate from one another. A syphon cannot

therefore be made to draw water from the depth of more than 33 feet, or mercury from more than 31 inches.

## CENTRIFUGAL PUMP.

259. *MLK* (Fig. 41.) is a tube of which the branch *ML* is horizontal, and *LK* immersed vertically in a fluid. The cylinder *LK* is moveable about its axis, and at *K* is a conical valve opening upwards.

The tube being filled with fluid and put in motion about the axis *KL*, a centrifugal force is generated in the portion *ML*, and the parts of the column *MLK* being held together by the atmospheric pressure on its extremities, a motion is thus communicated to the whole. And a stream of fluid is continually thrown off at *M*.

Let $a$ be the angular velocity $ML = a$, $CL = l$;

$$\therefore \text{(Art. 110.)} \int (X dx + Y dy + Z dz) = \tfrac{1}{2} a^2 a^2 - gl.$$

Therefore the tube being full of fluid, and the motion having become uniform,

$$\text{(Art. 132.)}\ p - p' = \tfrac{1}{2} a^2 a^2 - gl - \tfrac{1}{2} v^2.$$

Taking the integral from the surface $C$ (without the tube) where $a$, $l$, and $v$ are each evanescent, to the surface $M$. But at both these surfaces $p$ represents the unit of *atmospheric* pressure;

$$\therefore \tfrac{1}{2} a^2 a^2 - gl - \tfrac{1}{2} v^2 = 0;$$

$$\therefore v = \sqrt{a^2 a^2 - 2gl}.$$

260. If $H$ be the height of a column of the fluid whose weight is equal to the atmospheric pressure; $\sqrt{2gH}$ is the velocity with which it would ascend in the tube at $C$, provided a *perfect vacuum* were produced above that point. $\sqrt{2gH}$ is therefore a limit never exceeded by the actual velocity in the tube. Hence, we have the condition,

$$\sqrt{a^2 a^2 - 2gl} \not> \sqrt{2gh},$$

$$\text{and } a \not> \frac{\sqrt{2g\,(H+l)}}{a}.$$

261. Suppose a *given* quantity of fluid $PLQ$ to be contained in the tube. Let $CQ = z$, $PQ = c$, $PL = \rho$. Then since the value of $p$ is, at either extremity, the unit of atmospheric pressure; we have, by the general equation for motion in tubes, (Art. 165.),

$$\tfrac{1}{2} a^2 \rho^2 - g\,(l - z) - fc = 0.$$

$$\text{Now } l - z + \rho = c; \quad \therefore\ dz = d\rho;$$

$$\therefore\ \tfrac{1}{2} a^2 \rho^2 d\rho - g\,(l - z)\,dz - cf dz = 0,$$

$$\therefore\ \tfrac{1}{6} a^2 \rho^3 + \tfrac{1}{2} g\,(l - z)^2 - \tfrac{1}{2} cv^2 = C.$$

From which equation the motion may readily be determined.

## BARKER'S MILL.

262. A bent tube (Fig. 42.) consisting of two arms at right angles to each other is placed with one of them in a horizontal position, and made to be moveable about the axis of the other. The extremity of the horizontal arm is closed, and an aperture $P$ is made in its side, through which a fluid, supplied in a continued stream to the vertical arm, is allowed to escape. The reaction at $P$ gives motion to the system, (see Art. 46.)

Suppose the whole to have acquired an uniform motion, the influx being constant, and the surface of the fluid having attained a permanent position in $M$. Let $p$ be the unit of pressure on any point of the projection of the aperture $P$ on the opposite side of the tube. Let $PA = r$, $AM = z$, and let $K$, $K'$ and $k$ be sections of the tubes $AB$ and $AC$, and of the aperture $P$,

$V$ = the velocity of influx at $B$,

$V'$ = the velocity of the fluid in $AP$, in contact with the projection of $P$,

$v =$ the velocity of efflux at $P$,

$a =$ the angular velocity of the system,

$$p = \int (Xdx + Ydy + Zdz) - \tfrac{1}{2}v^2 + C,$$

and taking the integral from $M$ to $P$,

$$p - p_{,} = \tfrac{1}{2}a^2r^2 + gz - \tfrac{1}{2}(V'^2 - V^2),$$

where $p_{,}$ is the unit of atmospheric pressure. Now the *external* surface of the tube sustains the pressure of the atmosphere. The unit of *effective* pressure on the projection of $P$, is therefore $p - p_{,}$, or

$$\tfrac{1}{2}a^2r^2 + gz - \tfrac{1}{2}(V'^2 - V^2).$$

And, on the whole projection, the momentum of the pressure is

$$\left\{\tfrac{1}{2}a^2r^2 + gz - \tfrac{1}{2}(V'^2 - V^2)\right\} rk.$$

Suppose a force $Pg$ to be applied to the machine, acting with a momentum $Pgc$ about the axis $AB$. Now since the motion of the machine is *not accelerated* the sum of the momenta of the forces impressed upon it $= 0$;

$$\therefore \left\{\tfrac{1}{2}a^2r^2 + gz - \tfrac{1}{2}(V'^2 - V^2)\right\} rk - Pgc = 0 \ldots\ldots (1).$$

Also at the aperture $p = p_{,}$;

$$\therefore \tfrac{1}{2}a^2r^2 + gz - \tfrac{1}{2}(v^2 - V^2) = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Eliminating $\tfrac{1}{2}a^2r^2 + gz$, we obtain

$$\tfrac{1}{2}rk\left\{v^2 - V'^2\right\} - Pgc = 0 \ldots\ldots\ldots\ldots\ldots\ldots (3).$$

Now the *influx* being *given*, we have $kv = C$, also admitting the hypothesis of parallel sections $kv = K'V'$;

$$\therefore \tfrac{1}{2}krC^2\left\{\frac{1}{k^2} - \frac{1}{K'^2}\right\} - Pgc = 0;$$

$$\therefore P = \frac{krC^2}{2gc}\left\{\frac{1}{k^2} - \frac{1}{K'^2}\right\}.$$

The quantity $P$ manifestly *increases* as $k$ *diminishes*.

263. If the height $AB$ of the vertical cylinder be given $(=a)$, and the vessel be kept perpetually full, by equation (2),

$$\tfrac{1}{2}v^2\left(1-\frac{k^2}{K^2}\right)=\tfrac{1}{2}a^2r^2+ga;$$

Therefore by equation (3),

$$rk\left(\tfrac{1}{2}a^2r^2+ga\right)\left\{\frac{1-\dfrac{k^2}{K'^2}}{1-\dfrac{k^2}{K^2}}\right\}-Pgc=0.$$

Whence $a$ is given in terms of $P$, and conversely.

The other quantities being given, the weight $P$ will be raised with the greatest possible velocity, when the length of the arm $AC$

$$=\frac{3}{2}\cdot\frac{Pc}{ak}\cdot\frac{1-\dfrac{k^2}{K^2}}{1-\dfrac{k^2}{K'^2}}.$$

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264. We may *conceive* an instrument combining the principle of Barker's Mill, with that of the Centrifugal Pump. Suppose the tube $BAC$ (Fig. 42.) to be inverted. Let the extremity $B$ be immersed in a fluid, and let the machine be *filled* with the fluid and put in motion about the axis of $AB$.

The fluid will move up the tube and escape through the aperture $P$ on the principle of the Centrifugal Pump, and the re-action at $P$ will tend to perpetuate the motion on the principle of Barker's Mill.

Adopting the same notation, and *supposing* the machine to have attained an uniform motion, we shall obtain precisely as before,

$$rk\left(\tfrac{1}{2}a^2r^2 - ga\right)\left\{\frac{1-\dfrac{k^2}{K'^2}}{1-\dfrac{k^2}{K^2}}\right\} - Pgc = 0.$$

Observing that the weight of the column $AB$ tends to diminish the pressures.

## MONTGOLFIER'S HYDRAULIC RAM.

265. The Hydraulic or Water Ram is a contrivance for applying the momentum of a current of fluid to raise a portion of it.

The pipe $AB$ (Fig. 43.) communicates with a reservoir (or flowing stream) $A$. The point $B$ being as far as may be convenient below the level of the fluid in $A$. At $D$ is an aperture closed by a valve *which opens upwards* into an air vessel $E$. In this last is inserted the tube $EH$ through which the fluid is required to be raised. At $C$ is a second aperture *closing upwards* by means of a loaded valve. The water flows down the pipe $AB$, and escapes through the aperture $C$, until having acquired its maximum* (or permanent) velocity, the upward pressure of the stream on the inferior surface of the valve at this aperture becomes (by a proper adjustment of the loading) greater than the weight of the valve, and it closes. The momentum generated in the moving fluid is now wholly sustained by the coats of the tube. The valve $D$ is accordingly forced up, and the fluid is projected into the air vessel $E$ with the velocity it has acquired. The extremity of the tube $EH$ is now beneath the surface of the water in the air vessel, and the air being compressed above that surface, the water is made to ascend within the tube

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* It will be shown hereafter, that in some cases it is better that the fluid should *not* be allowed to acquire its maximum velocity before the valve at $C$ closes.

to a height dependant, upon the length of the tube $AB$, and the height of the surface of the fluid in $A$, above the point $B$. It is clear that after the ascent of the fluid to its greatest height in the air vessel, the valve $D$ will close, and, the motion in $ABC$ being destroyed, $C$ will open by reason of its weight. The motion will thus be *continued*, the fluid being alternately discharged at $C$, and thrown into the air vessel at $D$.

266. To simplify the theory of this instrument, let us suppose the valve $D$ to open (not into the air vessel $E$) but into an open vertical tube of the same diameter with $AB$. Let $P$ be any position of the surface of the fluid in this tube, $DP=z$, $ABD=a$, and let the altitude of the surface of the fluid in $A$ above the point $B$ be $h$. Now the maximum velocity acquired by the fluid in its efflux through $C$, will manifestly be that with which it would flow permanently if that aperture continued open. And this uniform velocity is represented (Art. 166.) by $\sqrt{2gh}$. Now when the valve $C$ closes, the fluid is projected up the tube $EH$, with this velocity. To determine its motion we have (Art. 165.)

$$p=D\left\{-gz-f(a+z)\right\}+C.$$

Hence neglecting the pressure resulting from the *motion* of the fluid at $A$, and taking the integral from $A$ to $P$,

$$0=g(h-z)-f(a+z);$$

$$\therefore\ vdv=fdz=\frac{g\,(h-z)\,dz}{a+z},$$

$$v^2=2g\left\{(a+h)\text{ h. l. }(a+z)-z\right\}+C.$$

$$\text{Now when } z=0,\ v=\sqrt{2gh}.$$

Since the fluid is projected up the tube $EH$ with that velocity;

$$\therefore\ v^2=2g\left\{(a+h)\text{ h. l. }\left(\frac{a+z}{a}\right)+(h-z)\right\}.$$

If we assume this expression $= 0$, the corresponding value of $z$ will be the greatest height to which the fluid is raised by the first impulse of the machine.

267. If we conceive the fluid to be discharged from the tube at the height $a'$ above $D$; it is evident that at every time of the opening of the valve $D$, the moving column $AD$ will encounter a quiescent column of fluid of the height $a'$. The common velocity after impact will by d'Alembert's Theorem be represented by $\dfrac{a\sqrt{2gh}}{a+a'}$.

Also if $f$ be the accelerating force at any period of the motion,

$$0 = g(h-a') - f(a+a');$$

$$\therefore \frac{dv}{dt} = -g\frac{(a'-h)}{a+a'}; \quad \therefore v = -g\frac{a'-h}{a+a'}t + C.$$

$$\text{Now when } t=0, \quad v = \frac{a\sqrt{2gh}}{a+a'};$$

$$\therefore v = \frac{a\sqrt{2gh} - g(a'-h)t}{a+a'}.$$

$$\text{Therefore when } v=0, \ t = \frac{a\sqrt{2h}}{(a'-h)\sqrt{g}}.$$

Which expression represents the whole time during which the fluid discharges itself from $H$.

268. To determine the *quantity* of the discharge; we have, calling $k$ a section of the tube,

$$\int kv\,dt = \frac{k}{a+a'}\int\{a\sqrt{2gh} - g(a'-h)t\}dt$$

$$= \frac{k}{a+a'}\left\{\frac{2a^2h}{a'-h} - \frac{a^2h}{a'-h}\right\} = \frac{a^2kh}{(a+a')(a'-h)}.$$

Taking the integral from

$$t = 0, \text{ to } t = \frac{a\sqrt{2h}}{(a'-h)\sqrt{g}}.$$

Again, to determine the motion of the fluid, when effluent through $C$. Let $a + a'' = AC$. If then $f$ be at any time the accelerating force on the column $AC$,

$$0 = gh - f(a + a'');$$

$$\therefore \frac{dv}{dt} = \frac{gh}{a + a''}, \quad \text{and } v = \frac{ght}{a + a''}.$$

Taking the integral from 0 to $t$.

Now when the motion becomes uniform, $v = \sqrt{2gh}$;

$$\therefore \frac{ght}{a + a''} = \sqrt{2gh}, \text{ and } t = \frac{a + a''}{\sqrt{\frac{1}{2}gh}}.$$

This value of $t$ represents the interval between each two successive projections of the fluid into the tube $EH$. If $T$ represent therefore any given time, $\frac{T\sqrt{\frac{1}{2}gh}}{a + a''}$ will represent the number of such projections in that time. Also $\frac{a^2kh}{(a+a')(a'-h)}$ is the quantity discharged during each;

$$\therefore \frac{Ta^2kh\sqrt{\frac{1}{2}gh}}{(a + a')(a + a'')(a' - h)},$$

represents the whole quantity discharged in the time $T$. It is evident that this quantity increases with $h$, which is essentially less than $a'$.

Also, the remaining quantities being given, it increases with $(a)$. For in this case it varies as

$$\frac{a^2}{(a + a')(a + a'')}, \text{ or as } \frac{1}{\left(1 + \frac{a'}{a}\right)\left(1 + \frac{a''}{a}\right)},$$

which last expression manifestly increases continually with $a$.

269. If $C$ lie on the opposite side of $D$, as is the more common construction of the instrument; the expression for the discharge in the given time $T$ will become

$$\frac{T\,(a-a'')^2\,kh\,\sqrt{\tfrac{1}{2}gh}}{(a+a')\,(a-a'')\,(a'-h)},$$

$$\text{or}\quad \frac{T\,(a-a'')\,kh\,\sqrt{\tfrac{1}{2}gh}}{(a+a')\,(a'-h)},$$

Now this expression is less than the preceding, the best construction is therefore that given in the figure.

270. If $v$ be at any time the velocity of the fluid effluent at $C$; $D\,(gh+\frac{1}{2}v^2)$ will be the corresponding unit of pressure on the inferior surface of the valve, and its maximum value will be $2Dgh$. Calling therefore $k$ the surface exposed to the action of the current, $2Dkgh$ will represent the maximum pressure on the valve, and the weight of the valve and loading must equal that of a super-incumbent column of fluid of the height $2h$.

271. In the above theory of the Hydraulic Ram, we have neglected to consider the *motion* of the fluid in the reservoir $A$. Suppose the section $(K)$ of this reservoir to be of *finite* dimensions, as compared with that $(k)$ of the pipe. Then, adopting the hypothesis of parallel sections, we shall have (Art. 257.) for the *time* of efflux through the valve $C$,

$$\frac{Nk}{\sqrt{2gh}}\ \text{h.l.}\left\{\frac{\sqrt{2gh}+v}{\sqrt{2gh}-v}\right\}.$$

Also for the *quantity* of each discharge at $H$,

$$N'k^2\ .\ \text{h.l.}\left\{\frac{2gh+v'^2}{2gh}\right\},$$

$v'$ representing the velocity which is, at first, communicated to the fluid in $EH$, and $N'$ the value of $N$ corresponding to the reservoir $A$ and the tubes $AD$ and $DH$.

Now, the *number* of discharges made in a given time, at $H$, varies inversely, as the time of each efflux at $C$. Therefore the whole discharge of the ram varies as

$$\frac{N'k\sqrt{2gh}}{N} \cdot \frac{\text{h. l.} \left\{\frac{2gh + v'^2}{2gh}\right\}}{\text{h. l.} \left\{\frac{\sqrt{2gh} + v}{\sqrt{2gh} - v}\right\}}.$$

$$\text{Now } v' = \frac{av}{a + a'}, \text{ very nearly.}$$

Also, if the height ($a'$) of the extremity $H$ of the tube be exceedingly great, as compared with $a$; $v'$ is exceeding small as compared with $v$, and therefore, *a fortiori*, as compared with $\sqrt{2gh}$;

$$\therefore \text{ discharge} \propto \frac{N'k}{N\sqrt{2gh}} \cdot \frac{v'^2}{\text{h. l.} \left\{\frac{\sqrt{2gh} + v}{\sqrt{2gh} - v}\right\}},$$

$$\propto \frac{N'ka^2}{N(a + a')^2\sqrt{2gh}} \cdot \frac{v^2}{\text{h. l.} \left\{\frac{\sqrt{2gh} + v}{\sqrt{2gh} - v}\right\}}.$$

Differentiating with regard to $v$, we shall obtain for the *maximum* discharge,

$$\text{h. l.} \frac{\sqrt{2gh} + v}{\sqrt{2gh} - v} - \frac{v\sqrt{2gh}}{2gh - v^2} = 0.$$

$$\text{Whence, writing } \frac{\sqrt{2gh} + v}{\sqrt{2gh} - v} = \Omega,$$

$$\Omega^2 - 4\Omega \,.\, \text{h. l. } \Omega - 1 = 0.$$

This equation is satisfied, by the value $\Omega = 8.226$;

$$\therefore v = \frac{7.8226}{9.8226}\sqrt{2gh} = \frac{4}{5}\sqrt{2gh} \text{ very nearly.}$$

# APPENDIX (A).

## ON THE VELOCITY OF SOUND, AND ON THE VIBRATIONS OF A CYLINDRICAL COLUMN OF AIR.

In treating of the motions of a fluid, constituted like the air, the only principle on which the calculation rests, in addition to the perfect mobility of the particles, is, that the density is always proportional to the pressure. But notwithstanding the fewness and simplicity of the principles, the general consideration of the motions of elastic fluids, presents great mathematical difficulties. In a few cases, however, in which the conditions of the problem are of a restricted nature, it is possible to arrive at results which may be compared with experiment. The case we are about to consider, is of the simplest kind: the fluid is supposed to be confined in a very slender cylindrical tube, the motions of the particles to be very small, and no extraneous force to act. Let the transverse section of the tube $=1$; let $ab$ (Fig. 68.) be a very small portion of the fluid at the distance $Oa=x$ from some fixed origin, the length $ab=\delta x$, and its density $=1+s$, the mean density being 1. Then the mass of $ab=(1+s)\delta x$. These values of $x$ and $s$ may be supposed to obtain at the end of a time $t$ reckoned from a fixed epoch. At the end of $t+\delta t$, let $ab$ be transferred to $a'b'$ with a velocity $v$, which may be considered uniform for the small time $\delta t$. Then

$$aa'=v\delta t, \quad Oa'=x+v\delta t, \quad Ob'=x+\delta x+v'\delta t;$$

$v'-v$ is the variation of $v$ from the point $a$ to the point $b$, the time being constant. Hence, by Taylor's theorem,

$$v'=v+\frac{dv}{dx}\delta x+\&c.$$

We have then, $Ob' = x + \delta x + \left(v + \frac{dv}{dx}\delta x\right)\delta t$, ultimately

$$Oa' = x + v\delta t;$$

$$\therefore\ a'b' = \delta x\left(1 + \frac{dv}{dx}\delta t\right).$$

Let $1 + s + \delta s$ be the density of $a'b'$. Then, because the mass of $ab$ is the same as that of $a'b'$,

$$(1+s)\,\delta x = (1 + s + \delta s)\left(1 + \frac{dv}{dx}\delta t\right)\delta x.$$

Hence, neglecting quantities of the second order,

$$(1+s)\frac{dv}{dx}\delta t + \delta s = 0, \quad \text{or} \quad \frac{\delta s}{\delta t} + (1+s)\frac{dv}{dx} = 0.$$

But $\frac{\delta s}{\delta t} = \left(\frac{ds}{dt}\right)$ ultimately. And $\left(\frac{ds}{dt}\right) = \frac{ds}{dt} + \frac{ds}{dx}\cdot\frac{dx}{dt}$, because $s$ is a function of $x$ and $t$. In the case we are considering, $s$ is a very small quantity. Therefore $\frac{ds}{dx}$ is very small; and $\frac{dx}{dt}$, which is equal to $v$, is also very small. Hence, neglecting $\frac{ds}{dx}\cdot\frac{dx}{dt}$, we obtain

$$\frac{ds}{dt} + (1+s)\frac{dv}{dx} = 0; \quad \text{or} \quad \frac{d\,.\,\text{h.l.}\,(1+s)}{dt} + \frac{dv}{dx} = 0 \ldots\ldots (A).$$

Let the pressure $= a^2 \times$ density. Now, by Art. 165, $dp = -(1+s)\,f\,dx$; the differentials being with respect to $x$ only, $t$ being constant. Hence,

$$\frac{a^2\,.\,d\,.\,(1+s)}{(1+s)\,dx} = -f = -\left(\frac{dv}{dt}\right), \text{ which is } = -\frac{dv}{dt} \text{ very nearly,}$$

for a reason similar to that above assigned with respect to $\left(\frac{ds}{dt}\right)$.

Consequently $\dfrac{a^2 \,.\, d^2 \,.\, \text{h.l.}\ (1+s)}{dx dt} = -\dfrac{d^2 v}{dt^2}$; and from ($A$),

$$\frac{a^2 \,.\, d^2\ \text{h.l.}\ (1+s)}{dx dt} = -\frac{a^2 d^2 v}{dx}.$$

$$\text{Therefore, } \frac{d^2 v}{dt} = a^2 \frac{d^2 v}{dx^2} \ldots\ldots\ldots\ldots\ldots (B).$$

Hence, integrating, (Lacroix, *Int. Calc.* Art. 139.)

$$v = F(x-at) + f(x+at) \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

$$\text{and } -\frac{dv}{dt} = a \,.\, F'(x-at) - a f'(x+at) = \frac{a^2 .d.\text{h.l.}\ (1+s)}{dx};$$

therefore, integrating with respect to $x$,

$a\,.\,\text{h.l.}\ (1+s) = F(x-at) - f(x+at) = as$, very nearly; (2)

and no function of $t$ is to be added, because

$$\frac{d\,.\,\text{h.l.}\ (1+s)}{dt} = -\frac{dv}{dt} \text{ from } (A).$$

Let us now consider what is indicated by the equations (1) and (2). The equation ($B$) will be satisfied if one of the functions $f$, be supposed to disappear, as we may convince ourselves by trial. Hence the equations

$$v = as = F(x - at)$$

point out a motion which is possible, and which it will be convenient to attend to in the first instance. Since $v = as$, if at a given instant an ordinate be erected at each point of the line $OMN$ (Fig. 69.), proportional to the difference between the density at the point and the mean density, these ordinates will also be proportional to the velocities, and the curve which bounds them will give at once the law of the density and that of the velocity. The positive ordinates correspond to velocities in the direction $OMN$ and to condensations, the negative to velocities in the contrary direction and to rarefactions. The state and motions of the particles

being, therefore, properly represented at the end of the time $t$, by some curve $Pqr$, the exact form of which for our present purpose, it is not necessary to know, let us enquire what will take place at the end of the time $t+\tau$. We shall have

$$v = as = F(x - a \,.\, \overline{t + \tau}).$$

Now if we describe two curves, the equation of one of which is $y = F(x - m)$, and that of the other $y = F(x - \overline{m + n})$, they must be exactly the same, but at different distances from the origin of $x$. An ordinate in the first will be less distant from the origin than the corresponding ordinate in the other, by the quantity $n$. Hence we infer, that the state of the particles at any distance $x + a\tau$ from the origin of co-ordinates in the case before us, at the end of the time $t + \tau$, is the same as the state of the particles at the distance $x$, at the end of $t$. The motion has consequently been *propagated* from the origin during the time $\tau$, through a space $a\tau$. As this is true whatever be $\tau$, the velocity of propagation is uniform and is equal to $a$. It is also independent of the magnitudes of $v$ and $s$. If $h =$ the height of the homogeneous atmosphere, $gh \times$ density $=$ pressure. Therefore $a^2 = gh$, and the velocity of propagation = that acquired by falling through half the height of the homogeneous atmosphere, as Newton first determined it to be.

Because the same quantity $a^2$ is equal to $\dfrac{\text{pressure}}{\text{density}}$, at every part of the atmosphere, it appears that the velocity of sound is the same in the higher regions, where the air is rarer, as it is at the surface of the earth. But the *intensity* of sound will be less as the air is rarer, since it must depend on the quantity of motion communicated by the disturbance, and the same disturbance (for instance, a stroke on a bell) will communicate a smaller quantity of motion to rare air than to air that is denser. It is observed that the report of a pistol on the tops of high mountains, is much less loud than in the plains below.

The velocity determined above for the propagation of sound, is that which would obtain in a medium in which

the pressure under *all* circumstances varies as the density, and no extraneous force acts. But experiment has shewn, that when the density of the air is *suddenly* altered, a developement of heat is produced, which either introduces a new force, or destroys the constancy of the quantity $\frac{\text{pressure}}{\text{density}}$. Now all the parts of the air through which motion is propagated, suffer a sudden alteration of density. Accordingly it is found that $a$, which is 925 feet per second, falls short of the actual velocity of propagation, which the experiments of the French Academicians ascertained to be 1105 feet per second. La Place has shewn by theoretical considerations (*Mec. Cel.* Liv. XII. Chap. 3.), that the effect of the developement of heat is taken into account by altering $a$ in a certain ratio. This ratio will therefore be that of 1105 to 925. It follows, that in the air as well as in the suppositious medium, the velocity of propagation is independent of the magnitude of the motions of the particles; and this is conformable to fact.

Just as we have treated the function $F$ by itself we might treat $f$ by itself, and we should obtain like results, excepting that the direction of propagation would be *towards* the origin of co-ordinates on the positive side. In general, therefore, the vibratory motions of the particles of a column of the fluid, may be resolved into the motions which result from two propagations obtaining simultaneously in opposite directions.

The equation $(B)$ will still be satisfied if each of the two functions contained in its integral, be resolved into as many others as we please, connected by the sign $+$ or $-$; that is, the equation is satisfied by

$$s = F_1(x-at) + F_2(x-at) - F_3(x-at) + \&c.$$
$$+ f_1(x+at) - f_2(x+at) + f_3(x+at) - \&c.$$

Now, as each of these terms may belong to a separate propagation, the interpretation of this analytical fact is, that a great number of propagations may obtain at the same time in the air, without interfering with each other, and that

many vibrations of its particles may *coexist.* The motion of a particle is the resultant of the several motions it would have, if each propagation took place without the others. We must conceive the simultaneous transmission of different sounds to be effected in this manner, in order to understand how it is that at a concert every ear is sensible of the effect of every instrument.

It was found that $v = as$ when the propagation is *from* the origin in the positive direction. Hence, since $v$ and $s$ are both positive or both negative at the same time, the condensed particles always move in the direction of propagation, the rarefied in the contrary direction. The same thing will appear from the equation $v = -as$, which applies to propagation in the negative direction. This will help us to explain how it is that the same disturbance, for instance, the motion of a small body forwards in the fluid, will produce propagations simultaneously in opposite directions. For the fluid must be just as much condensed at one part by the disturbance, as rarefied at another. But the body impresses both on the condensed and rarefied particles, a motion in the direction in which itself moves. Therefore the rarefied portion will produce propagation in a different direction from the condensed.

Let us now attend to the forms of the functions $F$ and $f$. These will be given by the given nature of the disturbances. Let $abcd$ (Fig. 70.) be a cylindrical tube containing the fluid, $mn$ a diaphragm placed transverse to the axis, just fitting the interior, and capable of moving in the direction of the axis. If the diaphragm be made to move with a velocity which is very small compared to the velocity of propagation, the particles immediately in contact with the side looking in the direction of the motion, will be condensed proportionally to its velocity; the particles on the other side will be in the same proportion rarefied. For the equations $v = as$ and $v = -as$ must apply to these cases. These condensations and rarefactions will be propagated from $m$ with the uniform velocity $a$. Suppose the diaphragm to move in the direction $ma$ from rest to rest again, in the time $\tau$, through a small space

which may be neglected in comparison of the distance over which the motion is propagated in the same time. Take $em = a\tau$; then if $em$ represent the whole time of the motion of the diaphragm, and an ordinate $pq$ be erected proportional to its velocity at a time from the commencement of its motion represented by the abscissa $ep$, the locus of $q$ will be a curve, which may be called the *type* of the wave generated by the diaphragm. The same curve inverted, as $mq'e'$, will be the type of the wave propagated in the other direction. Since

$$v = as = F(x - at),$$

the curve $eqm$ is expressed analytically by the arbitrary function, and we therefore see that the form of the function is given by the given mode of the disturbance. From the time that the diaphragm comes to rest, all the subsequent motion will be indicated by the curve $eqm$, moving with the velocity $a$ in the direction $ma$, without undergoing any change of form, for there are no data whereby a change of form could be determined. The particles it successively reaches, all pass through the same states of velocity and condensation as those in immediate contact with the diaphragm. That the line $eqm$ may be a *portion* of a curve, for instance, a segment of a circle, has been proved by Lagrange, who shews generally in the second volume of the Miscellanea Taurinensia, that the consecutive values of $v$ and $s$ are not necessarily subject to the law of continuity. Hence also it is not necessary that the line between $e$ and $m$ should be continuous; it may even consist of two straight lines as $ef$, $fm$. If the diaphragm moved uniformly for any length of time, the line $eqm$ would become a straight line parallel to $am$, indicating that a *stream* would be produced: all the particles in motion will be condensed proportionally to their velocity; for the equation $v = as$ must apply to this case.

If the diaphragm go on moving backwards and forwards so that its oscillations shall be isochronous, it will generate a series of alternate condensations and rarefactions, which are proper for giving to the ear the sensation of a musical note. For the only condition required for producing a musical sound, is, that the waves which strike the ear recur at regular intervals. The *pitch* of the note depends on the number of waves which strike

in a given time: the greater the number the higher the note. If the number exceed a certain limit the note will be too high to be perceived, and on the other hand, the ear is unable to appreciate the note when the number falls short of a certain limit: these limits are different for different ears. It is not necessary that the type of all the waves should be the same, or the oscillations of the diaphragm be all performed in exactly the same manner; provided all occupy the same time the note will remain the same, but will be clearer the more regular the waves are. What has been said will serve to convey a general idea of the manner in which the *pitch* of a musical note is determined by the vibrations of a solid in the air, as for instance, the reed in organ-pipes. It is probable that the particular manner of the vibration, the position of the vibrating body with respect to the parts of the instrument to which it is attached, and the vibrations of the instrument itself, all go to determine the *timbre* of the note, or that quality by which we distinguish the same note on different instruments.

But there are cases in which musical sounds are produced without the intervention of a vibrating solid. If a uniform stream of air be blown across the mouth of an open cylindrical tube, or obliquely against the edge of its mouth, a musical note will in many cases be produced. (See Biot, Traitè de Physique, Tom. II. Chap. ix.) Now if this stream were confined in a straight canal, we should know from what has been said in the last paragraph but one, that the condensation would be proportional to its velocity. And without entering into the consideration of motion in space of three dimensions, we may infer that in any case the density of the stream will differ from the mean density. But the air within the tube, being kept at rest by its sides, will be of mean density. Hence two portions of air of different densities will be contiguous to each other, and consequently motion must ensue. From the manner of directing the stream, it is plain that the column of air in the tube, undergoes but a very small permanent motion of translation, even when the mouth opposite to that at which the disturbance is made is open, as in the common flute with the finger holes stopped, and none at all, if the opposite mouth be closed, as in

Pan's pipes. Hence the column will be made to vibrate backwards and forwards, and experience shews that the vibrations may be such as to give rise to musical notes. Here then we have an instance in which the vibrations are caused by the action of the parts of the fluid on one another, and in which also there is no extraneous accelerative force. Hence the precise nature of the vibrations must be determined by referring to the integral of equation ($C$), and endeavouring to ascertain, prior to all hypothesis about the mode of disturbance, the particular form of its arbitrary functions: for this equation has been investigated solely in reference to the action of the parts of the fluid on each other. We shall succeed in doing this in the following manner.

It has been shewn prior to any hypothesis about the mode of disturbance, that each of the functions $F$ and $f$, in the integral of ($C$) will satisfy it independently of the other, and that one applies to a propagation in the positive direction, the other to a propagation in the contrary direction. When therefore both occur in the integral at the same time, it is allowable to suppose, as a particular case, that the propagations they indicate are exactly equal to each other. In such a case there must be one point at least, at which the particles go through the same series of velocities by reason of the two propagations, but in opposite directions. At this point therefore the resulting velocity must be 0, independently of the time. Let $l$ be its distance from the origin of co-ordinates, and for $at$ put $z$.

Then $F(l-z)-f(l+z)=0$, whatever be $z$. ($a$)

$\therefore$ by Taylor's theorem,

$$F(l)-f(l)-\{F'(l)+f'(l)\}z+\{F''(l)-f''(l)\}\frac{z^2}{2}-\&c.=0;$$

$$\begin{array}{lr} \text{Hence } F(l)-f(l)=0, & (1) \\ F'(l)+f'(l)=0, & (2) \\ F''(l)-f''(l)=0, & (3) \\ \&c. & \&c. \end{array}$$

These equations can determine nothing about the value of $l$, which must remain arbitrary, as the origin of co-ordinates is arbitrary. They must be satisfied, therefore, by a consideration of the values and forms of the *functions themselves.* We shall

satisfy at once (1), (3), (5), &c. by making $f$ the same as $F$. Equation ($a$) then becomes $F(l-z)-F(l+z)=0$, and shews that $F(l)$ is a maximum or minimum, and that the values of the function $F$ at equal distances on each side of the maximum or minimum value are equal. Hence also $F'(l)=0$, and the equation (2) is satisfied. But besides this we must have

$$F'''(l)=0,\ \ F^{\text{v}}(l)=0,\ \ \&\text{c.}$$

That is, the same value of $x$ which makes $F(x)$ a maximum or minimum, makes all the odd differential coefficients disappear. This condition immediately conducts us to a trigonometrical function, and the simplest that presents itself is $y=m\sin x$, which satisfies all the required conditions. The only other mode in which any of the equations (1), (2), (3), &c. may be identically satisfied, is by making $f$ the same as $-F$. This supposition verifies at once (2), (4), (6), &c. and ($a$) becomes

$$F(l+z)+F(l-z)=0, \text{ or } F(l+z)=-F(l-z).$$

This being true whatever be $z$, shews, by making $z=0$, that $F(l)=0$, and equation (1) is satisfied. But we must also have $F''(l)=0$, $F^{\text{iv}}(l)=0$, &c. We have therefore to find a curve, such that the same value of $x$ which makes $y=0$, causes all the even differential coefficients to disappear, and so disposed about a point at which it cuts the axis, that the ordinates at equal distances on each side of this point shall be equal with opposite signs. These conditions will be fulfilled in the same equation $y=m\sin x$. Moreover the required conditions are satisfied in the most general manner by $y=m\sin x+m'\sin 3x+m''\sin 5x+\&\text{c.}$ on account of the unlimited number of terms. But, as we have seen, this equation points out a motion which is the resultant of a great number of motions of the kind indicated by $y=m\sin x$. This last may therefore be called the *primary* form of the arbitrary function, and is that which it was required to find. Let $\lambda$ be the common distance between the points at which the curve cuts the axis: then $y=m\lambda\sin\dfrac{\pi x}{\lambda}$.

Before entering upon the consideration of the vibrations of the column of air in a stop-pipe, let us consider generally the mode in which a series of aerial waves are *reflected*. Conceive two motions exactly alike to be propagated in opposite directions along a cylindrical tube. There must be a point at which the velocities are the same and in the same order, in virtue of the two propagations, but in opposite directions. At this point therefore the particles will be at rest. The motion will in no respect be changed, if an indefinitely thin rigid partition be placed at right angles to the axis of the tube just where the particles are stationary. The fluid will be divided into two separate columns in each of which the motions will be the same as before. But plainly the particles in one column cannot be affected by a disturbance made in the other. Hence the effect of such disturbance is supplied by reflection at the partition. It thus appears that the obstacle gives rise to a series of reflected waves exactly like the incident waves, and that the particles in contact with the reflecting body do not move.

Suppose now a series of waves of the primary type to be generated in the manner we have before mentioned, at the open end $A$ (Fig. 71.) of a tube closed at $B$, and to be propagated from $A$ towards $B$. At $B$ they will be reflected, will return to $A$, and there issue out into the circumambient fluid. After reflection two waves, whose types are $cb$, $c'b'$, exactly equal, will meet, and in consequence at some point $m$ the velocity will be always equal to 0. Let $t$ be reckoned from the time at which $c$ and $c'$ were simultaneously at $m$. Then $cm = at$; and if $mp = x$, $pq = y$,

$$y = m\lambda \sin\frac{\pi}{\lambda}(x + at).$$

$$\text{Also if } pr = y',\ y' = m\lambda \sin\frac{\pi}{\lambda}(at - x).$$

$$\text{Hence } v = y - y' = m\lambda\left(\sin\frac{\pi}{\lambda}.\overline{x+at} + \sin\frac{\pi}{\lambda}.\overline{x-at}\right)$$

$$= 2m\lambda \sin\frac{\pi x}{\lambda}\cos\frac{\pi a t}{\lambda},$$

$$\text{and } as = y + y' = m\lambda\left(\sin\frac{\pi}{\lambda}(x+at) - \sin\frac{\pi}{\lambda}(x-at)\right)$$

$$= 2m\lambda \cos\frac{\pi x}{\lambda}\sin\frac{\pi a t}{\lambda}.$$

As the particles at the closed end must necessarily be at rest, we may reckon $x$ from $B$ towards $A$, and date from an instant at which the condensation at $B$ is 0. Then $v$ will always $= 0$ where $x = n\lambda$, and $as$ will always $= 0$ where $x = (n + \frac{1}{2})\lambda$. The points obtained by putting $n = 0, 1, 2, 3$, &c. are in the first case called *nodes*, in the other, *loops*. Experience shews that when a musical note is sounded, the density of the fluid at $A$ is always the mean. This will take place if $A$ be the position of a loop. Hence if $AB = l$, we must have $l = (n + \frac{1}{2})\lambda$, or $\lambda = \dfrac{2l}{2n+1}$. Let $n = 0$, then $\lambda = 2l$. In this case the lowest note, the *fundamental note*, is sounded. If we call it 1, the others may be called 3, 5, 7, &c. being inversely as $\lambda$ the breadth of a wave. It is found in fact that 1, 3, 5, &c. are the only notes that can be sounded. If we suppose the end $B$ to be removed, and the tube to be prolonged to $A'$, so that $BA' = BA$, the wave instead of being reflected at $B$ will go on to $A'$, and pass out there just as in the other case it passed out at $A$. Hence the disturbance being the same, the note of a tube open at both ends is the same as that of a tube of half the length closed at one end:—an inference which experiment confirms. It is found that the series of notes is 1, 2, 3, 4, &c. or that they are such that the breadth $\lambda$ of a wave is an aliquot part of the length of the tube. The reason of this fact is not satisfactorily understood, but probably may be traced to the vibrations of the tube itself, which would conspire with waves of this series, and interfere with any other.

# APPENDIX (B.)

ON THE GENERAL EQUATION. Art. 132.

If we take the line along which the integration ($A$), Article 131 is to be made, always in the direction of the motion of the particles through which it passes or in the opposite direction; the quantities $dx$, $dy$, $dz$, taken (according to the conditions of equation $A$) from one point in space to another, are equivalent to $\pm dx$, $\pm dy$, $\pm dz$ taken in reference to the *motion* of each particle. Now $dx$, $dy$, $dz$ being *thus* taken $\int(\phi dx + \phi' dy + \phi'' dz)$ is shown Art. 156.) to be equivalent to $\int \frac{dv}{dt} ds$;

$$\therefore\ p = P \mp D \int \frac{dv}{dt} ds\, *.$$

Suppose the motion to be uniform, and the line of particles taken as above to occupy the path of a *given* particle $\mu$. Now the integral is to be taken at a *given time*, from one point to another in this line, and with respect to the different particles of it. But, on the assumed hypothesis of uniform motion, the integral thus taken, is the same as though it were taken with regard to the *same* particle $\mu$, when at

* This result may be deduced at once by considering the whole fluid mass (held in equilibrium by the application of the forces lost) to become solid, excepting only a line of particles taken as above, of whose length $ds$ is an element.

$v$ representing the velocity in magnitude and direction, $\frac{dv}{dt}$ will under every circumstance represent the accelerating force, also in magnitude and direction. The equation is therefore perfectly general.

*different times* it occupies the different points in its path. The integral $\int \frac{dv}{dt}\, ds$ therefore, in fact, represents the sum of the effective accelerating forces on the particle $\mu$ at the different periods of its motion, each multiplied by a corresponding element $ds$ of the space it describes.

$v$ and $s$ are both functions of $t$, and therefore of one another;

$$\therefore \frac{dv}{dt} = \frac{dv}{ds}\,\frac{ds}{dt} = v\,\frac{dv}{ds}; \quad \therefore \int \frac{dv}{dt} = \int v\,\frac{dv}{ds}\,ds = \int v\,dv = \tfrac{1}{2}v^2;$$

$$\therefore p = P \mp \tfrac{1}{2} D v^2 + C.$$

The ambiguous sign is manifestly introduced in passing from the conditions of the equilibrium to those of the motion.

According to the above theory the negative sign should be replaced by the sign $\mp$ in equation ($e$), Art. 206.

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*Note on Article* 156. The quantities $dx$, $dy$, $dz$ are assumed in this Article to have reference to the *motion* of the fluid, as explained in Article 132.

## APPENDIX (C.)

### ON THE EFFLUX OF FLUIDS.

LET a fluid be supposed to flow through an aperture of finite dimensions, in the base of a vessel, towards which its sides contract, and let us consider the descent of an element contained at any given time by two horizontal sections of the fluid.

Since in being made to occupy a lower position in the vessel, the element is reduced in magnitude, it is clear that a portion of the fluid it contains is extruded from it in its

descent, and left to occupy a higher position in the vessel. The particles among which the velocity of descent is thus partially destroyed (and which manifestly occur towards the extremity of each section) being thus forced amongst those of the immediately superior section, a still further disturbance and deflexion is the result. And thus towards the boundary of every section similarly taken in the fluid, a retardation is produced, at once by the continual contraction of its dimensions, and the partial quiescence of the inferior fluid.

And further it is evident that the particles at any given time occupying a given section of the fluid, will in their further descent form themselves into a curved surface, whose curvature is continually variable with the time.

In contact therefore with the sides of the vessel, there is formed a fluid mass in which motion is partly or wholly destroyed; and, if there be any portion of the fluid in which the hypothesis of continual descent in the same horizontal section obtains, it is bounded by a surface other than that of the sides of the vessel. It would seem that such a portion of fluid exists near the axis of the aperture presenting a slender column, whose section varies as the squares of the radii of curvature, at the vertices of the surfaces into which the descending sections of the fluid successively form themselves. This column is clearly bounded by a surface variable in form with the time. The following experiments confirm the observations we have made above.

*Experiment* 1. If into a prismatic vessel containing a fluid which escapes through a small aperture in its base; there be thrown minute particles whose specific gravity is somewhat greater than that of the fluid; they will be observed to descend *vertically* until they reach within three radii of the aperture; after which they will converge from every side towards it, describing curved lines manifestly convex to the axis of the vessel. Thus the moving fluid will form in the vicinity of the aperture a conoid rapidly converging to it, and having the section of the vessel for its superior base, and three radii of the aperture for its height. This conoid is called the gorge. The small quantity of fluid surrounding

the gorge remains stagnant near the bottom of the vessel. The tendency of all particles towards the aperture by a conoidal funnel is equally apparent, whether it be made in the bottom or the sides of the vessel.

*Experiment* 2. If there be poured on the surface of the fluid a stratum of oil or other coloured liquid of a less specific gravity than itself; so soon as that stratum has reached within the distance of three radii of the aperture, the coloured liquid will force its way through the fluid to reach it, and the gorge will be clearly seen converging to the aperture, and convex to the axis of the vessel.

*Experiment* 3. When the aperture is made in a thin substance, the particles preserve during a short space after they have passed it, the oblique and converging direction in which they approached it. The *jet* is thus rapidly contracted near its commencement, and there is formed outside of the vessel a second conoid which may be considered a continuation of that within it.

This external conoid is called the *vena contracta*, and its extreme or least section, *the* section of the *vena contracta.*

The ratio of the section of the *vena contracta* to that of the aperture is represented in all cases by the quantity 0.625.

It is clear that the *vena contracta* will produce the same variation in the efflux as though it formed a continuation of the vessel itself, and that the efflux does in fact take place at its extreme section. In all those formulæ into which there enters the symbol $k$, representing the section of the aperture at which the efflux takes place, it must be understood that it does not represent the section of the aperture *in the vessel*, but that section multiplied by the decimal 0.625. Where the motion takes place from one fluid into another, as in the case of communicating vessels, the ratio of the section of the *vena contracta* to that of the aperture is no longer the same. It has not as yet been ascertained.

*To determine the surfaces of equal pressure in a moving fluid.*

We have if the motion be uniform,

$$gz - \tfrac{1}{2}v^2 = \text{constant}.$$

Admitting the hypothesis of parallel sections, and conceiving the interior surface of the vessel to be symmetrical about a vertical axis which passes through the aperture, and such that all its horizontal sections may be similar, we shall have for the equation to any section through the axis of a vessel of equal pressure,

$$gz - \frac{c}{y^4} = C.$$

The above is the equation to an hyperbolic curve of the fourth order between the asymptotes.

The *vena contracta* is evidently a surface of equal pressure.

## APPENDIX (D).

### ON THE RESISTANCE OF FLUIDS.

In the theory of resistance given in Article 171, no account is taken of that variation in the pressure which results from the *disturbance* at the *posterior* surface of the plane. This hypothesis, which is that usually adopted, is erroneous. A variation is manifestly produced in the pressure of the fluid, as well by the motion *behind* as by that *before* the plane.

Let the plane $xy$ be taken *beneath* the fluid, then adopting the notation of Art. 192, and considering the motion of a particle from the point where the disturbance *commences*, to the anterior surface of the plane; we have, since the motion tends to *increase* the co-ordinates,

$$p - p_{,} = - gD\,(z - z_{,}) - \tfrac{1}{2}D\,(v^2 - v_{,}^2).$$

Again, considering the motion of a particle from the point where the disturbance *terminates* to the *posterior* surface, we have, since the motion tends to diminish the co-ordinates,

$$p - p_{,,} = - gD\,(z - z_{,,}) + \tfrac{1}{2}D\,(v^2 - v_{,,}^2).$$

Subtracting these equations,

$$p_{,} - p_{,,} = - gD\,(z_{,} - z_{,,}) + Dv^2 - \tfrac{1}{2}D\,(v_{,}^2 - v_{,,}^2),$$

$$\Sigma p_{,}\mu - \Sigma p_{,,}\mu = - gD\Sigma\,(z_{,} - z_{,,})\,\mu + D\Sigma v^2\mu - \tfrac{1}{2}D\Sigma\,(v_{,}^2 - v_{,,}^2)\,\mu.$$

Now the integral $-gd\Sigma\,(z_{,} - z_{,,})\,\mu$, taken with regard to that portion of the plane, *both* surfaces of which lie beneath the surface of the fluid, evidently $=0$, and taken with regard to the remainder, it is shown, Art. 193, to equal $\dfrac{mDv^4}{2g}$.

If therefore we assume the velocity of the fluid *in contact* with the plane, to be, on both sides of it the same (or $v_{,} = v_{,,}$), we shall have for the whole pressure, $(\Sigma p_{,}\mu - \Sigma p_{,,}\mu)$, tending to produce motion,

$$\frac{mDv^4}{2g} + MDv^2.$$

Throughout this theory of resistance we have supposed each particle of fluid to lose its velocity on coming in contact with the anterior surface, and to re-acquire it by a continual acceleration on the opposite side of the plane.

This seems to amount to the hypothesis, that the plane whilst it gradually destroyed all motion in the particles as they approached it, should when they came in contact with it, present no further obstacle to their progress, an hypothesis manifestly opposed to the facts of the case. The fluid brought in contact with the plane having lost all its velocity in the direction of its motion, collects in a quiescent state before it, and presents to the action of the current a fluid surface essentially different from that of the plane, and tending obviously to vary the pressure upon it.

The following theory is by La Grange *. Suppose a circular plane to be opposed transversely to the current of a stream, then will a conoidal mass of *quiescent* fluid collect itself before it, bounded by a correspondent hollow conoid of *moving* fluid. Suppose every portion of this last fluid to move with the same velocity, each transverse section of the conoidal shell, into which it has formed itself will then be of the same area. Let $K$ represent this area, $R$ the radius of curvature of any point of the curve, whose revolution generates the surface bounding of the quiescent conoid, $x$ the perpendicular distance of this point from the *vertex*, and $y$ its distance from the axis of the conoid, $v$ the velocity of the stream.

Now let us consider a section of the conoidal shell perpendicular to its surface. The pressure exerted by each of the moving particles in a direction perpendicular to that of its motion, (i. e. the centrifugal force) is represented by $\frac{v^2}{R}$, and, considering the thickness of the shell as small, the whole pressure thus generated on an annulus of the quiescent conoid, is represented by $\frac{v^2 K ds}{R}$. Referred to an unit of surface this becomes $\frac{v^2 K ds}{R 2\pi y ds}$. Now this unit of pressure is the same throughout the whole surface of the conoid, and is transferred through the medium of the quiescent fluid to each unit of the plane. If therefore, we call $P$ the unit of pressure on the plane, we have

$$P = \frac{K v^2}{2\pi R y}.$$

$$\text{Now } R = \frac{-dy}{d\left(\frac{dx}{ds}\right)};$$

* *Mem. de Turin*, 1784, 1785.

$$\therefore\ 2\pi P y dy = - K v^2 d\left(\frac{dx}{ds}\right);$$

$$\therefore\ P\pi y^2 = C - K v^2 \left(\frac{dx}{ds}\right) \text{...........}(1),$$

$\frac{dx}{ds}$ is the cosine of the angle made by a tangent to the generating curve with the axis of the conoid. At the vertex therefore this cosine equals unity. Let $\phi$ represent this angle at the point, where the curve meets the plane. Now at the base of the conoid $P\pi y^2$ represents the whole pressure upon the plane. Taking the integral therefore from the vertex of the conoid to its base, we obtain for the whole pressure on the plane, the expression

$$K v^2 (1 - \cos\phi) \text{....................}(2).$$

From equation (1), we obtain

$$x = \int \frac{(C - P\pi y^2)\, dy}{\sqrt{K^2 v^4 - (C - P\pi y^2)^2}}.$$

By the integration of which expression the nature of the generating curve will be determined.

## APPENDIX (E).

### On the Note to Art. 125.

Since the temperature diminishes in arithmetical progression as the altitude increases in arithmetical progression; it is clear that the corresponding variations of the temperature and altitude are to one another in a constant ratio;

$$\therefore\ t' - t^0 \propto z = Cz;$$

$$\therefore \int \frac{dz}{(1+at^0)(a+z)^2} = \int \frac{dz}{\{1+a(t'-Cz)\}(a+z)^2}$$

$$= \int \frac{(1-at'+Caz)\,dz}{(a+z)^2} \text{ nearly};$$

$\therefore$ &c.................................&c.

The above method is evidently preferable to that given in the commencement of the note.

## APPENDIX (F).

### On Art. 206.

By Art. 156. we have

$$p = P \mp D \int \frac{dV}{dt}\, ds,$$

$V$ being the velocity of any particle, and $ds$ the space it describes; and the integral being taken along an irregular line, drawn continually in the direction of the motion of the particles through which it passes, or in the opposite direction.

Now, $V$ varies by reason of the increment $dt$ in time, of the motion of the particle whose velocity it represents, and the variation $ds$ of its position in space;

$$\therefore \frac{dV}{dt} = \left(\frac{dV}{dt}\right) + \left(\frac{dV}{ds}\right)\frac{ds}{dt}$$

$$= \left(\frac{dV}{dt}\right) + V\left(\frac{dV}{ds}\right);$$

$$\therefore \int \frac{dV}{dt}\, ds = \int \left(\frac{dV}{dt}\right) ds + \int V \left(\frac{dV}{ds}\right) ds$$

$$= \int \left(\frac{dV}{dt}\right) ds + \int V dV$$

$$= \int \left(\frac{dV}{dt}\right) ds + \tfrac{1}{2} V^2.$$

The constant, which is a function of $t$, being supposed to be included in the integral $\int\left(\frac{dV}{dt}\right)ds$. On the whole we have, therefore,

$$p = P \mp D\int\left(\frac{dV}{dt}\right)ds \mp \tfrac{1}{2}DV^2.$$

In the integral $\mp\int\left(\frac{dV}{dt}\right)ds$, $\mp ds$ represents the elementary space actually described by any particle of the line along which the integration is made; and the sign $\mp$ is taken according as that motion is in the direction in which the line is *measured*, or in the opposite direction. If we alter the hypothesis, and suppose $ds$ to be an element of the line itself, the ambiguous sign will disappear, since $ds$, will in this case, become essentially positive. We shall thus obtain

$$p = P - D\int\left(\frac{dV}{dt}\right)ds \mp \tfrac{1}{2}DV^2.$$

Adopting the notation of Art. 206,

$$V^2 = u^2 + v^2 + w^2;$$

and differentiating with respect to $t$,

$$V\left(\frac{dV}{dt}\right) = u\left(\frac{du}{dt}\right) + v\left(\frac{dv}{dt}\right) + w\left(\frac{dw}{dt}\right);$$

$$\therefore \frac{ds}{dt}\left(\frac{dV}{dt}\right) = \frac{dx}{dt}\left(\frac{du}{dt}\right) + \frac{dy}{dt}\left(\frac{dv}{dt}\right) + \frac{dz}{dt}\left(\frac{dw}{dt}\right);$$

$$\therefore \left(\frac{dV}{dt}\right)ds = \left(\frac{du}{dt}\right)dx + \left(\frac{dv}{dt}\right)dy + \left(\frac{dw}{dt}\right)dz$$

$$\ldots\ldots\ldots\ldots = d\left(\frac{d\phi}{dt}\right) \text{ by Art. 206;}$$

$$\therefore \int\left(\frac{dV}{dt}\right)ds = \left(\frac{d\phi}{dt}\right);$$

$$\therefore p = P - D\left(\frac{d\phi}{dt}\right) \mp \tfrac{1}{2}DV^2;$$

which agrees with equation (*e*), Art. 206, except that the last term is not there affected by the *ambiguous sign.* The introduction of this sign involves a question of some difficulty and of very considerable importance. It is opposed to the theory of D'Alembert, and to the exposition of that theory given in Chap. VII.: under these circumstances he is most anxious to show that the innovation has not been made lightly or inadvertently.

It may perhaps put the subject in a clearer light to consider the equation $A$, Art. 131, in its original form without substituting (in the beginning of the operation) for $dx$, $dy$, $dz$ which are there supposed to be taken from one point in space to another, their equivalents $\pm dx$, $\pm dy$, $\pm dz$ which represent the elementary *motions* of a given particle.

From the equation in its original form, we shall obtain precisely as before

$$p = P - D\int \frac{dV}{dt}\, ds.$$

$$\text{Now } \frac{dV}{dt} = \left(\frac{dV}{dt}\right) + V\left(\frac{dV}{ds}\right).$$

Where the differentiation is made with reference to the increment of time $dt$, and the variation *of position* $ds$; $ds$ here then represents an element of the *motion* of a particle in the line along which the integration is made, and the partial differential coefficient $\left(\frac{dV}{dt}\right)$ is taken on this hypothesis. But in the integral $\int \frac{dV}{dt}\, ds$, $ds$ represents an element (not of the motion, but) of the *line* of integration.

$$\text{Now, } \int \frac{dV}{dt}\, ds = \int \left(\frac{dV}{dt}\right) ds + \int V\left(\frac{dV}{ds}\right) ds.$$

In $\int V\left(\frac{dV}{ds}\right) ds$, $\left(\frac{dV}{ds}\right)$ represents therefore, the partial

differential coefficient of $V$ with respect to the space described by a given particle, whilst $ds$ represents, not an element of that space, but a given positive element of the line itself.

In the present form of the expression, the integration cannot therefore be effected. To complete it, we must substitute for the *element of the line ds*, its equivalent, *the element of the motion* $\mp ds$; and we shall thus have

$$\int V\left(\frac{dV}{ds}\right)(\pm ds) = \pm\int V\left(\frac{dV}{ds}\right)ds = \pm\tfrac{1}{2}V^2;$$

$$\therefore\ p = P - D\int\left(\frac{dV}{dt}\right)ds \mp \tfrac{1}{2}DV^2.$$

It may here be as well to observe, that the pressure *on any point of a* moving *fluid, is not* necessarily the same in *every* direction.

## APPENDIX (G).

NOTE ON ART. 157.

In this Article $v$ and $\kappa$ are functions of $t$; differentiating $\frac{vk}{\kappa}$ with respect to $t$, we therefore, in fact, differentiate the velocity $v$ *generally*; and $\frac{dv}{dt}$ represents, as it ought, not the partial differential coefficient of $v$ with respect to $t$, but simply the limiting ratio of the increments of $v$ and $t$.

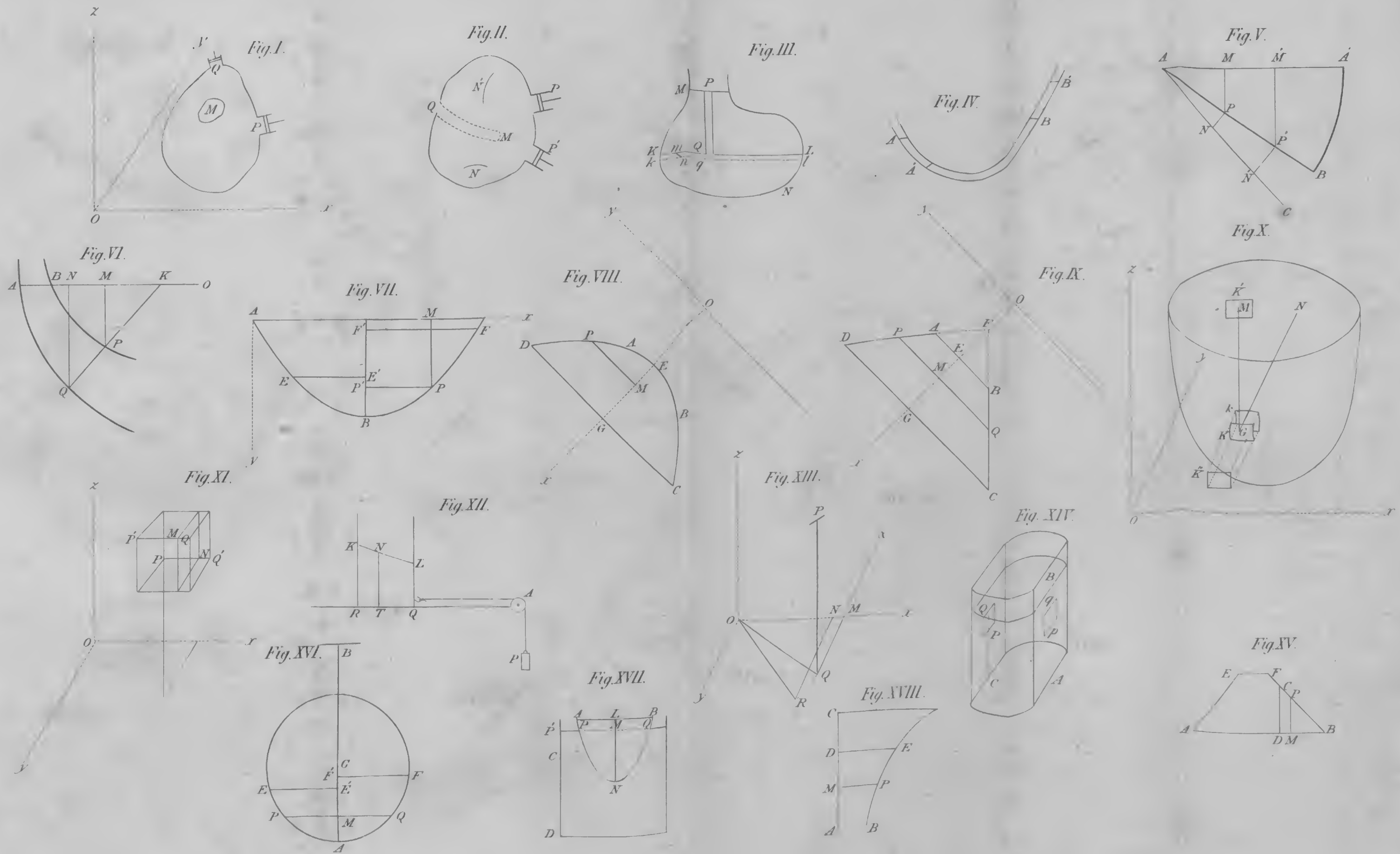


Fig. I. Fig. II. Fig. III. Fig. IV. Fig. V. Fig. VI. Fig. VII. Fig. VIII. Fig. IX. Fig. X. Fig. XI. Fig. XII. Fig. XIII. Fig. XIV. Fig. XV. Fig. XVI. Fig. XVII. Fig. XVIII.

Fig. XIX.

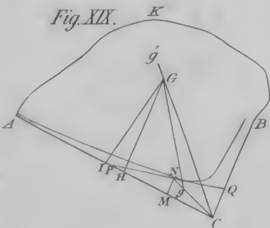


Fig. XX.

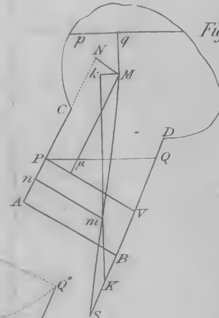


Fig. XXI.

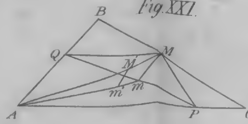
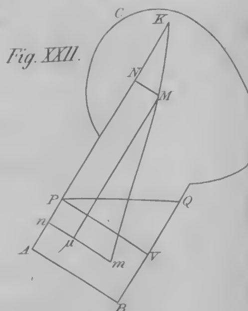


Fig. XXII.



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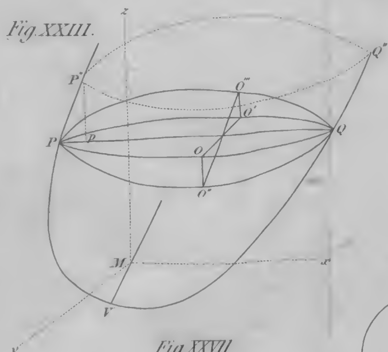

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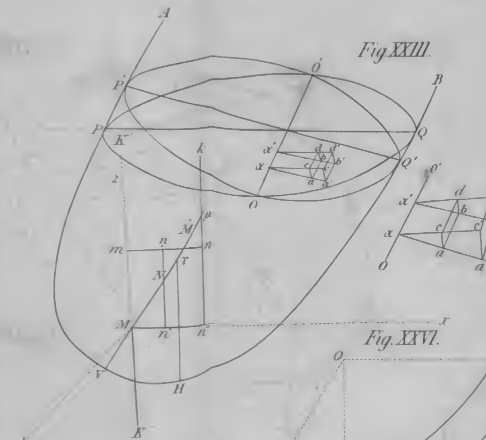

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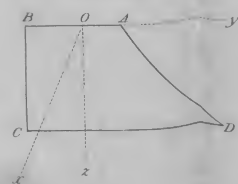

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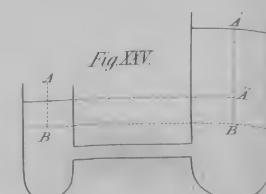

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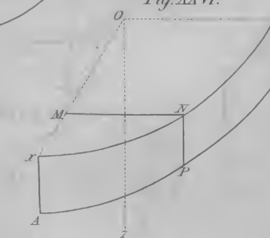

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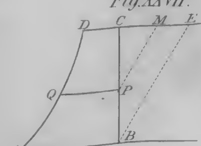


Fig. XXX.

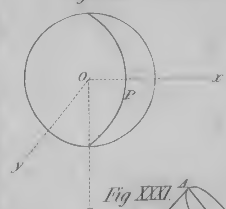
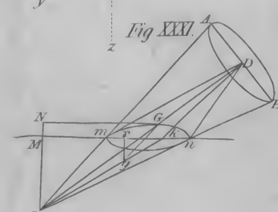


Fig. XXVIII.

Fig. XXIX.

Fig. XXXI.



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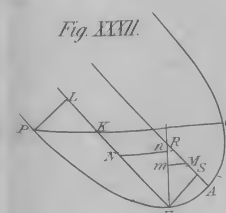

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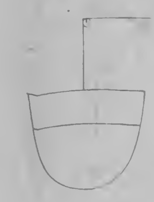

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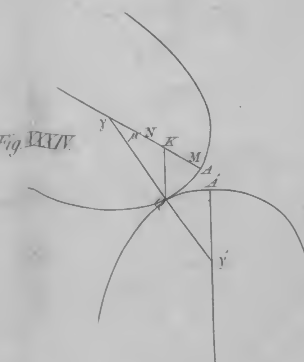

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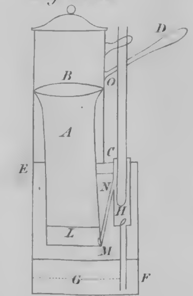


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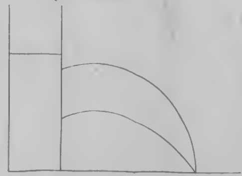

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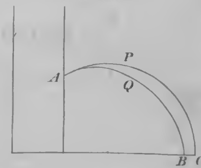


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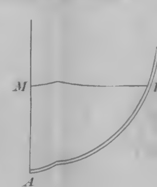


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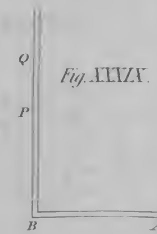
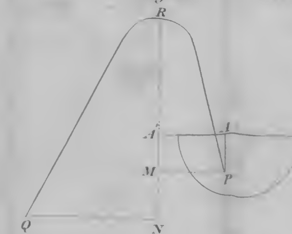



Fig. XL.

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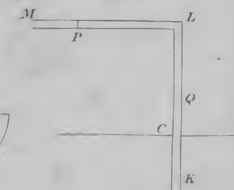


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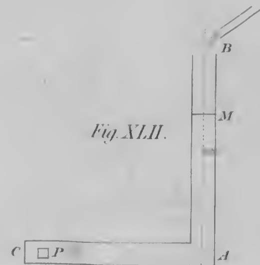


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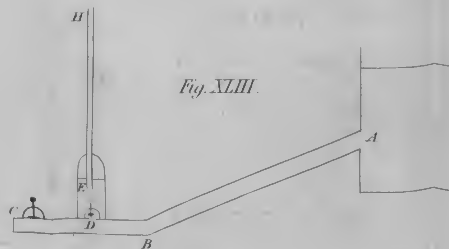


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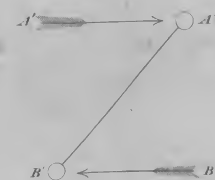


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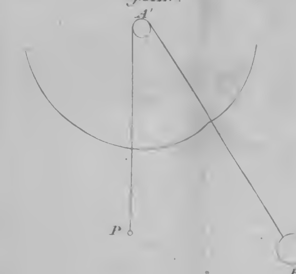


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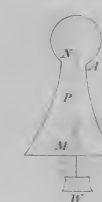


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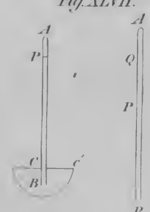


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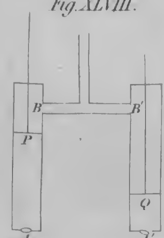


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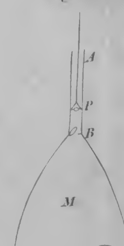



Fig. L.

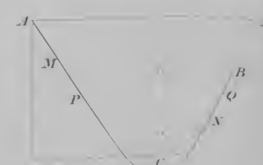



Fig. LI.

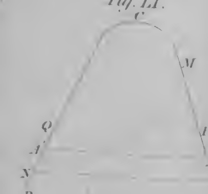
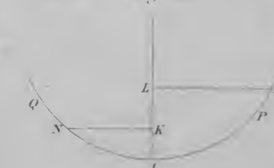



Fig. LII



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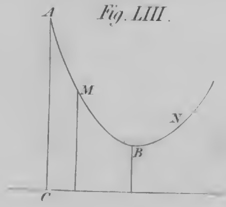


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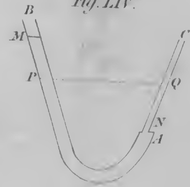


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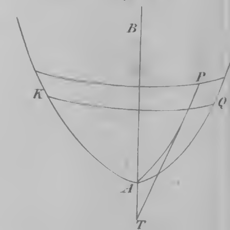


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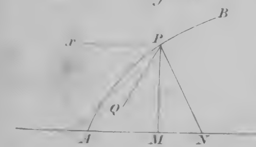


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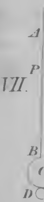


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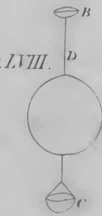


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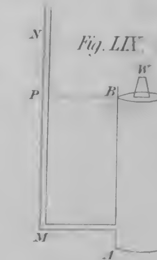


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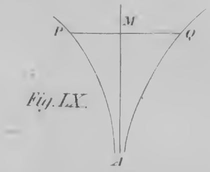


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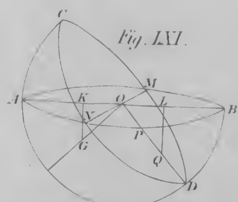


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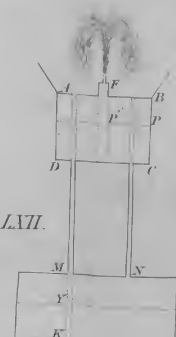


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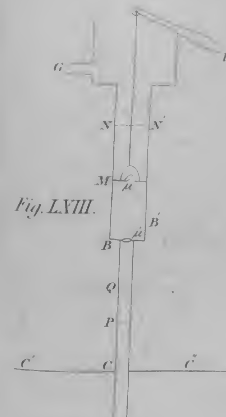


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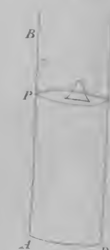


Fig. LXV.

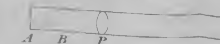


Fig. LXVI.

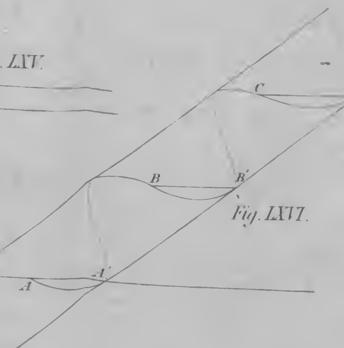


Fig. LXVII.

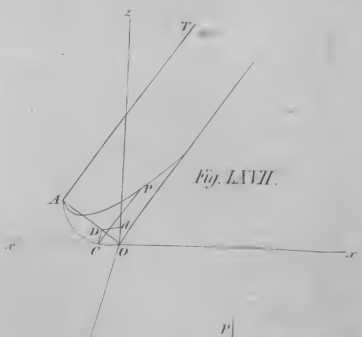


Fig. LXXII.

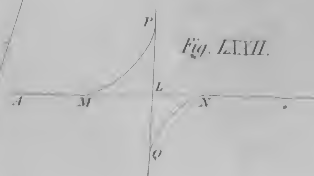


Fig. LXVIII.

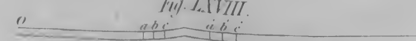


Fig. LXIX.

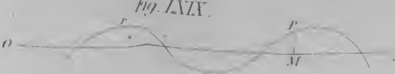


Fig. LXX.

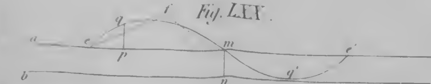


Fig. LXXI.